Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 8

Due Date: 23.06.2014

Exercise 1:

Let X and Y be integral k-schemes of finite type and let $f : X \to Y$ be a dominant morphism of finite type. Let $d = \dim X - \dim Y = \mathrm{trdeg}_{K(Y)} K(X)$.

(i) Let $y \in Y$. Show that $\dim f^{-1}(y) \geq d$.

(ii) Show that there is an open dense subset $U \subset Y$ such that $\dim f^{-1}(y) = d$ for all $y \in U$.
(*Hint: Reduce to the case* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$. *Then there are* $T_1, \dots, T_d \in B$ *which form a transcendental basis of* $K(X)$ *over* $K(Y)$. *Now consider the (fibers of the) morphism* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A[T_1, \dots, T_d]$.)

(iii) For $x \in X$ set $h(x) = \max\{\dim Z\}$, where Z runs over all irreducible components of $f^{-1}(f(x))$ containing x. Show that $\{x \in X \mid h(x) \geq e\}$ is closed in X for all $e \in \mathbb{Z}$.
(*Hint: Use induction on* $\dim X$.)

Remark: If f is proper, then it follows that the map $y \mapsto \dim f^{-1}(y)$ is upper semi-continuous.

Exercise 2:

(i) Let $f : X \to Y$ be a flat morphism of k-schemes of finite type. Show that for all $x \in X$ and $y = f(x) \in Y$ one has

$$\dim_x f^{-1}(y) = \dim_x X - \dim_y Y,$$

where $\dim_x X = \dim \mathcal{O}_{X,x}$ (and similarly for the other dimensions).
(*Hint: First reduce to the case* $Y = \mathrm{Spec}\, \mathcal{O}_{Y,y}$. *Then use induction on* $\dim Y$ *and Krull's principal ideal theorem. Use flatness of* $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ *to show that the image of a non-zero divisor is a non-zero divisor.*)

(ii) Give an example of a proper morphism $f : X \to Y$ such that

  (a) the fiber dimension of f is not constant (hence f is not flat).

  (b) f is flat and f is smooth over $f^{-1}(U)$ for some open dense subset $U \subset X$ but there is a point $y \in Y$ such that $f^{-1}(y)$ is connected and not irreducible (hence f is flat but not smooth).

Exercise 3:

Let $\mathcal{C}$ be an abelian category. Show that in $\mathcal{C}$ arbitrary pull-backs and push-outs exists.

Exercise 4:

Prove the snake lemma in an arbitrary abelian category.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`