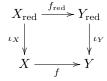
Algebraic Geometry I Exercise Sheet 8 Due Date: 12.12.2013

Exercise 1:

(i) Let (X, \mathcal{O}_X) be a scheme and $Z \subset X$ be a (locally) closed subset. Show that there is a unique structure of a closed subscheme (Z, \mathcal{O}_Z) on Z such that (Z, \mathcal{O}_Z) is reduced.

If we choose Z = X in (i), then we write X_{red} for the resulting scheme and call it the *reduced* scheme underlying X.

(ii) Let $f: X \to Y$ be a morphism of schemes. Show that f induces a morphism $f_{red}: X_{red} \to Y_{red}$ such that the diagram



commutes, where $\iota_X : X_{red} \hookrightarrow X$ and $\iota_Y : Y_{red} \hookrightarrow Y$ are the canonical closed immersions.

Exercise 2:

- (i) Let $f : X \to Y$ be a morphism of schemes and let $j : Z \hookrightarrow Y$ be an immersion. Then f factors over j if and only if $f(X) \subset j(Z)$ and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ factors over $\mathcal{O}_Y \to j_*\mathcal{O}_Z$.
- (ii) Let $f: X \to Y$ be morphism of schemes and let $Z \subset Y$ be a subscheme. Assume in addition that X is reduced. Show that f factors over the inclusion $Z \hookrightarrow Y$ if and only if $f(X) \subset Z$.

Exercise 3:

Let k be a field. Describe the fibers of the following morphisms. Which fibers are reduced? Which fibers are irreducible? Try to draw pictures of the situation.

- (i) Spec $k[X, Y]/(XY 1) \to$ Spec k[X]
- (ii) Spec $k[X, Y]/(X^2 Y^2) \to \operatorname{Spec} k[X]$
- (iii) Spec $k[X, Y]/(X^2 + Y^2) \to \operatorname{Spec} k[X]$
- (iv) Spec $k[X, Y]/(XY) \to \operatorname{Spec} k[X]$
- (v) Spec $k[X, Y, Z]/(Y^2 XZ) \rightarrow \text{Spec } k[X]$
- (vi) Spec $k[T, U, V, W]/((U+T)W, (U+T)(U^3 + U^2 + UV^2 V^2)) \rightarrow \text{Spec } k[T]$

(In all cases the morphism Spec $B \to \text{Spec } A$ is induced by the obvious ringhomomorphism $A \to B$.)

Exercise 4:

- (i) Let K be a field and let \bar{K} denote an algebraic closure of K. Let L be an algebraic extension of K. Compute Spec $L \times_{\text{Spec } K} \text{Spec } \bar{K}$.
- (ii) Let R be a ring. Show that $\mathbb{A}_R^n \times_{\operatorname{Spec} R} \mathbb{A}_R^m \cong \mathbb{A}_R^{n+m}$ and $\mathbb{A}_R^n \times_{\operatorname{Spec} R} \operatorname{Spec} S \cong \mathbb{A}_S^n$ for all R-algebras S.
- (iii) Let k be an algebraically closed field. Show that the product $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ embeds into \mathbb{P}^3_k as a closed subscheme which is isomorphic to the quadric $V_+(T_0T_3 T_1T_2)$.

(Hint: Show that in homogenous coordinates (on closed points) the immersion is given by $(x_0:x_1), (y_0:y_1) \mapsto (x_0y_0:x_0y_1:x_1y_0:x_1y_1)$ and use an affine cover of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.)

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