Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 3

Due Date: 07.11.2013

Exercise 1:

Let $(X, \mathcal{O}_X)$ be a prevariety over k and $Z \subset X$ be an irreducible closed subset. For $U \subset Z$ open we set

$$\mathcal{O}_Z(U) = \{f : U \to k \mid \forall x \in U \ \exists x \in V \subset X \text{ open and } g \in \mathcal{O}_X(V) \text{ such that } f|_{U \cap V} = g|_{U \cap V}\}.$$

(i) Show that the inclusion $Z \hookrightarrow X$ defines a morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$.

(ii) Show that for $z \in Z$ the induced map on local rings $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is surjective.

(iii) Assume that X is affine. Then Z is an irreducible affine algebraic set. Show that the space with functions associated to Z agrees with $(Z, \mathcal{O}_Z)$. Especially $\mathcal{O}_X(X) \to \mathcal{O}_Z(Z)$ is surjective.

(iv) Give an example of a prevariety X and a closed sub-prevariety Z such that $\mathcal{O}_X(X) \to \mathcal{O}_Z(Z)$ is not surjective.

Exercise 2:

Let $\mathfrak{a} \subset k[T_0, \dots, T_n]$ be a homogenous radical ideal and let $X = V_+(\mathfrak{a}) \subset \mathbb{P}^n$ be the corresponding projective variety. Set $S = k[T_0, \dots, T_n]/\mathfrak{a} = \bigoplus_{d \geq 0} S_d$ and $S_+ = \bigoplus_{d > 0} S_d$, where $S_d = k[T_0, \dots, T_n]_d/(\mathfrak{a} \cap k[T_0, \dots, T_n]_d)$.

(i) Show that the points of X correspond bijectively to the homogenous maximal ideals of S not equal to S_+.

(ii) Show that the closed subsets of X correspond bijectively to the homogenous radical ideals of S not equal to S_+.

(iii) For $f \in S$ homogenous set $D_+(f) = \{x \in X \mid f(x) \neq 0\}$. Show that the $D_+(f)$ are open and form a basis of the topology of X.

(iv) For $f \in S$ homogenous set $S_{(f)} = \{g \in S_f \mid g \text{ is homogenous of degree } 0\}$. Show that there are bijections

$$\begin{aligned}\{x \in D_+(f)\} &\longleftrightarrow \{S_+ \neq \mathfrak{m} \subset S \text{ homogenous maximal ideal such that } f \notin \mathfrak{m}\} \\ &\longleftrightarrow \{\text{maximal ideals } \mathfrak{m} \subset S_{(f)}\}\end{aligned}$$

inducing a homeomorphism of $D_+(f)$ with the affine variety defined by $S_{(f)}$.

In fact the open subvariety $D_+(f)$ is affine and isomorphic to the affine variety defined by the k-algebra $S_{(f)}$. We have seen this already explicitly for $X = \mathbb{P}^n$ and $f = T_i$.

Exercise 3:

Let $X \subset \mathbb{P}^n$ be a projective variety and let $f : X \to \mathbb{P}^m$ be a map of sets. Show that f is a morphism if and only of there is a covering $X = \bigcup_{i\in I} U_i$ and for each U_i there exist homogenous polynomials $f_0^{(i)}, \dots, f_m^{(i)} \in k[T_0, \dots, T_n]$ such for each $x = (x_0 : \dots : x_n) \in U_i$ at least one of the $f_j^{(i)}(x)$ does not vanish and

$$f((x_0 : \dots : x_n)) = (f_0^{(i)}(x) : \dots : f_m^{(i)}(x)) \in \mathbb{P}^n.$$

Exercise 4:

Let $n, d \geq 1$ and set $N = \binom{n+d}{n} - 1$. We denote by $M_0, \dots, M_N \in k[T_0, \dots, T_n]$ the monomials of degree d. Define a map

$$\vartheta_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

by $x = (x_0 : \dots : x_n) \mapsto (M_0(x) : \dots : M_N(x))$. This map is called the *d-fold Veronese embedding.*

(i) Show that ϑ_d is a map of projective varieties.

(ii) Show that the image $\vartheta_d(\mathbb{P}^n)$ is a closed irreducible subset $V_+(\mathfrak{a}) \subset \mathbb{P}^N$ for some homogenous ideal $\mathfrak{a} \subset k[T_0, \dots, T_N]$ and that ϑ_d identifies $\mathbb{P}^n$ with the projective variety $V_+(\mathfrak{a})$.

(iii) Let $f \in k[T_0, \dots, T_n]$ be homogenous of degree d. Show that $\vartheta_d(V_+(f)) \subset \mathbb{P}^N$ is the intersection of $\vartheta_d(\mathbb{P}^n)$ with a linear subspace of $\mathbb{P}^N$.

(iv) Let $Y \subset \mathbb{A}^3$ denote the affine algebraic set $\{(t, t^2, t^3) \mid t \in k\}$ and let $\bar{Y} \subset \mathbb{P}^3$ be the closure of Y under the embedding $(x_1, x_2, x_3) \mapsto (1 : x_1 : x_2 : x_3)$ of $\mathbb{A}^3$ in $\mathbb{P}^3$. Describe the ideal $\mathfrak{a} = \{f \in k[T_0, \dots, T_3] \mid f(x) = 0 \text{ for all } x \in \bar{Y}\}$ and show that $\bar{Y}$ is identified (for a suitable choice of coordinates) with the image of the 3-fold Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`