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# Substitution in First-Order Formulas: Elementary Properties[1]

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**Summary.** This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article introduces the basic concepts of substitution of a variable for a variable in a first-order formula. The contents of this article correspond to Chapter III par. 8, Definition 8.1, 8.2 of Ebbinghaus, Flum, Thomas.

MML Identifier: `SUBSTUT1`.

The terminology and notation used here are introduced in the following articles: [15], [7], [17], [18], [4], [12], [1], [14], [2], [11], [8], [6], [3], [9], [19], [5], [10], [13], and [16].

## 1. Preliminaries

For simplicity, we follow the rules: $a$, $b$ are sets, $i$, $k$ are natural numbers, $x$, $y$ are bound variables, $P$ is a $k$-ary predicate symbol, $l_1$ is a variables list of $k$, $l_2$ is a finite sequence of elements of Var, and $p$ is a formula.

The functor vSUB is defined by:

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(Def. 1) $\mathrm{vSUB} = \mathrm{BoundVar} \dot{\rightarrow} \mathrm{BoundVar}$.

One can check that vSUB is non empty.

A CQC-substitution is an element of vSUB.

Let us note that vSUB is functional.

In the sequel $S_1$ is a CQC-substitution.

Let us consider $S_1$. The functor ${}^{@}S_1$ yielding a partial function from BoundVar to BoundVar is defined as follows:

(Def. 2) ${}^{@}S_1 = S_1$.

Next we state the proposition

(1) If $a \in \mathrm{dom}\, S_1$, then $S_1(a) \in \mathrm{BoundVar}$.

Let $l$ be a finite sequence of elements of Var and let us consider $S_1$. The functor $\mathrm{CQC\text{-}subst}(l, S_1)$ yields a finite sequence of elements of Var and is defined as follows:

(Def. 3) $\mathrm{len}\, \mathrm{CQC\text{-}subst}(l, S_1) = \mathrm{len}\, l$ and for every $k$ such that $1 \leq k$ and $k \leq \mathrm{len}\, l$ holds if $l(k) \in \mathrm{dom}\, S_1$, then $(\mathrm{CQC\text{-}subst}(l, S_1))(k) = S_1(l(k))$ and if $l(k) \notin \mathrm{dom}\, S_1$, then $(\mathrm{CQC\text{-}subst}(l, S_1))(k) = l(k)$.

Let $l$ be a finite sequence of elements of BoundVar. The functor ${}^{@}l$ yielding a finite sequence of elements of Var is defined by:

(Def. 4) ${}^{@}l = l$.

Let $l$ be a finite sequence of elements of BoundVar and let us consider $S_1$. The functor $\mathrm{CQC\text{-}subst}(l, S_1)$ yields a finite sequence of elements of BoundVar and is defined as follows:

(Def. 5) $\mathrm{CQC\text{-}subst}(l, S_1) = \mathrm{CQC\text{-}subst}({}^{@}l, S_1)$.

Let us consider $S_1$ and let $X$ be a set. Then $S_1 \restriction X$ is a CQC-substitution.

One can verify that there exists a CQC-substitution which is finite.

Let us consider $x$, $p$, $S_1$. The functor $\mathrm{RestrictSub}(x, p, S_1)$ yielding a finite CQC-substitution is defined by:

(Def. 6) $\mathrm{RestrictSub}(x, p, S_1) = S_1 \restriction \{y : y \in \mathrm{snb}(p) \wedge y$ is an element of $\mathrm{dom}\, S_1 \wedge y \neq x \ \wedge\ y \neq S_1(y)\}$.

Let us consider $l_2$. The functor $\mathrm{BoundVars}(l_2)$ yielding an element of $2^{\mathrm{BoundVar}}$ is defined as follows:

(Def. 7) $\mathrm{BoundVars}(l_2) = \{l_2(k) : 1 \leq k \ \wedge\ k \leq \mathrm{len}\, l_2 \ \wedge\ l_2(k) \in \mathrm{BoundVar}\}$.

Let us consider $p$. The functor $\mathrm{BoundVars}(p)$ yielding an element of $2^{\mathrm{BoundVar}}$ is defined by the condition (Def. 8).

(Def. 8) There exists a function $F$ from WFF into $2^{\mathrm{BoundVar}}$ such that

(i) $\mathrm{BoundVars}(p) = F(p)$, and

(ii) for every element $p$ of WFF and for all elements $d_1$, $d_2$ of $2^{\mathrm{BoundVar}}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \emptyset_{\mathrm{BoundVar}}$ and if $p$ is atomic, then $F(p) = \mathrm{BoundVars}(\mathrm{Args}(p))$ and if $p$ is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = d_1$ and if $p$ is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = d_1 \cup d_2$ and if $p$ is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = d_1 \cup \{\mathrm{Bound}(p)\}$.

One can prove the following propositions:

(2) $\mathrm{BoundVars}(\mathrm{VERUM}) = \emptyset$.

(3) For every formula $p$ such that $p$ is atomic holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{Args}(p))$.

(4) For every formula $p$ such that $p$ is negative holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{Arg}(p))$.

(5) For every formula $p$ such that $p$ is conjunctive holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{LeftArg}(p)) \cup \mathrm{BoundVars}(\mathrm{RightArg}(p))$.

(6) For every formula $p$ such that $p$ is universal holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{Scope}(p)) \cup \{\mathrm{Bound}(p)\}$.

Let us consider $p$. One can check that $\mathrm{BoundVars}(p)$ is finite.

Let us consider $p$. The functor $\mathrm{DomBoundVars}(p)$ yielding a finite subset of $\mathbb{N}$ is defined as follows:

(Def. 9) $\mathrm{DomBoundVars}(p) = \{i : \mathrm{x}_i \in \mathrm{BoundVars}(p)\}$.

In the sequel $f_1$ denotes a finite CQC-substitution.

Let us consider $f_1$. The functor $\mathrm{Sub\text{-}Var}(f_1)$ yields a finite subset of $\mathbb{N}$ and is defined as follows:

(Def. 10) $\mathrm{Sub\text{-}Var}(f_1) = \{i : \mathrm{x}_i \in \mathrm{rng}\, f_1\}$.

Let us consider $p$, $f_1$. The functor $\mathrm{NSub}(p, f_1)$ yields a non empty subset of $\mathbb{N}$ and is defined as follows:

(Def. 11) $\mathrm{NSub}(p, f_1) = \mathbb{N} \setminus (\mathrm{DomBoundVars}(p) \cup \mathrm{Sub\text{-}Var}(f_1))$.

Let us consider $f_1$, $p$. The functor $\mathrm{upVar}(f_1, p)$ yielding a natural number is defined as follows:

(Def. 12) $\mathrm{upVar}(f_1, p) = \min \mathrm{NSub}(p, f_1)$.

Let us consider $x$, $p$, $f_1$. Let us assume that there exists $S_1$ such that $f_1 = \mathrm{RestrictSub}(x, \forall_x p, S_1)$. The functor $\mathrm{ExpandSub}(x, p, f_1)$ yielding a CQC-substitution is defined by:

(Def. 13) $\mathrm{ExpandSub}(x, p, f_1) = \begin{cases} f_1 \cup \{\langle x, \mathrm{x}_{\mathrm{upVar}(f_1, p)} \rangle\}, & \text{if } x \in \mathrm{rng}\, f_1, \\ f_1 \cup \{\langle x, x \rangle\}, & \text{otherwise.} \end{cases}$

Let us consider $p$, $S_1$, $b$. The predicate $b = \mathrm{PQSub}(p, S_1)$ is defined as follows:

(Def. 14) If $p$ is universal, then $b = \mathrm{ExpandSub}(\mathrm{Bound}(p), \mathrm{Scope}(p), \mathrm{RestrictSub}(\mathrm{Bound}(p), p, S_1))$ and if $p$ is not universal, then $b = \emptyset$.

The function QSub is defined as follows:

(Def. 15) $a \in \mathrm{QSub}$ iff there exist $p$, $S_1$, $b$ such that $a = \langle \langle p, S_1 \rangle, b \rangle$ and $b = \mathrm{PQSub}(p, S_1)$.

## 2. Definition and Properties of the Formula – Substitution – Construction

In the sequel $e$ denotes an element of vSUB.

We now state the proposition

(7)(i) $[\!:$ WFF, vSUB $:\!]$ is a subset of $[\!: [\!: \mathbb{N}, \mathbb{N} :\!]^*, \text{vSUB} :\!]$,

(ii) for every natural number $k$ and for every $k$-ary predicate symbol $p$ and for every list of variables $l_1$ of the length $k$ and for every element $e$ of vSUB holds $\langle \langle p \rangle \frown l_1, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$,

(iii) for every element $e$ of vSUB holds $\langle \langle \langle 0, 0 \rangle \rangle, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$,

(iv) for every finite sequence $p$ of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and for every element $e$ of vSUB such that $\langle p, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$ holds $\langle \langle \langle 1, 0 \rangle \rangle \frown p, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$,

(v) for all finite sequences $p$, $q$ of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and for every element $e$ of vSUB such that $\langle p, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$ and $\langle q, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$ holds $\langle \langle \langle 2, 0 \rangle \rangle \frown p \frown q, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$, and

(vi) for every bound variable $x$ and for every finite sequence $p$ of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and for every element $e$ of vSUB such that $\langle p, \text{QSub}(\langle \langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle \frown p, e \rangle) \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$ holds $\langle \langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle \frown p, e \rangle \in [\!: \text{WFF}, \text{vSUB} :\!]$.

Let $I_1$ be a set. We say that $I_1$ is QC-Sub-closed if and only if the conditions (Def. 16) are satisfied.

(Def. 16)(i) $I_1$ is a subset of $[\!: [\!: \mathbb{N}, \mathbb{N} :\!]^*, \text{vSUB} :\!]$,

(ii) for every natural number $k$ and for every $k$-ary predicate symbol $p$ and for every list of variables $l_1$ of the length $k$ and for every element $e$ of vSUB holds $\langle \langle p \rangle \frown l_1, e \rangle \in I_1$,

(iii) for every element $e$ of vSUB holds $\langle \langle \langle 0, 0 \rangle \rangle, e \rangle \in I_1$,

(iv) for every finite sequence $p$ of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and for every element $e$ of vSUB such that $\langle p, e \rangle \in I_1$ holds $\langle \langle \langle 1, 0 \rangle \rangle \frown p, e \rangle \in I_1$,

(v) for all finite sequences $p$, $q$ of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and for every element $e$ of vSUB such that $\langle p, e \rangle \in I_1$ and $\langle q, e \rangle \in I_1$ holds $\langle \langle \langle 2, 0 \rangle \rangle \frown p \frown q, e \rangle \in I_1$, and

(vi) for every bound variable $x$ and for every finite sequence $p$ of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and for every element $e$ of vSUB such that $\langle p, \text{QSub}(\langle \langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle \frown p, e \rangle) \rangle \in I_1$ holds $\langle \langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle \frown p, e \rangle \in I_1$.

Let us mention that there exists a set which is QC-Sub-closed and non empty.

The non empty set QC-Sub-WFF is defined as follows:

(Def. 17) QC-Sub-WFF is QC-Sub-closed and for every non empty set $D$ such that $D$ is QC-Sub-closed holds QC-Sub-WFF $\subseteq D$.

In the sequel $S$, $S'$, $S_2$, $S_3$, $S_1'$, $S_2'$ are elements of QC-Sub-WFF.

Next we state the proposition

(8) There exist $p$, $e$ such that $S = \langle p, e \rangle$.

Let us note that QC-Sub-WFF is QC-Sub-closed.

Let $P$ be a predicate symbol, let $l$ be a finite sequence of elements of Var, and let us consider $e$. Let us assume that $\mathrm{Arity}(P) = \mathrm{len}\, l$. The functor $\mathrm{SubP}(P, l, e)$ yields an element of QC-Sub-WFF and is defined as follows:

(Def. 18) $\mathrm{SubP}(P, l, e) = \langle P[l], e\rangle$.

We now state the proposition

(9) Let $k$ be a natural number, $P$ be a $k$-ary predicate symbol, and $l_1$ be a list of variables of the length $k$. Then $\mathrm{SubP}(P, l_1, e) = \langle P[l_1], e\rangle$.

Let us consider $S$. We say that $S$ is sub-verum if and only if:

(Def. 19) There exists $e$ such that $S = \langle \mathrm{VERUM}, e\rangle$.

Let us consider $S$. Then $S_{\mathbf{1}}$ is an element of WFF. Then $S_{\mathbf{2}}$ is an element of vSUB.

The following proposition is true

(10) $S = \langle S_{\mathbf{1}}, S_{\mathbf{2}}\rangle$.

Let us consider $S$. The functor $\mathrm{SubNot}(S)$ yields an element of QC-Sub-WFF and is defined as follows:

(Def. 20) $\mathrm{SubNot}(S) = \langle \neg(S_{\mathbf{1}}), S_{\mathbf{2}}\rangle$.

Let us consider $S$, $S'$. Let us assume that $S_{\mathbf{2}} = S'_{\mathbf{2}}$. The functor $\mathrm{SubAnd}(S, S')$ yields an element of QC-Sub-WFF and is defined by:

(Def. 21) $\mathrm{SubAnd}(S, S') = \langle S_{\mathbf{1}} \wedge S'_{\mathbf{1}}, S_{\mathbf{2}}\rangle$.

In the sequel $B$ denotes an element of [: QC-Sub-WFF, BoundVar :].

Let us consider $B$. Then $B_{\mathbf{1}}$ is an element of QC-Sub-WFF. Then $B_{\mathbf{2}}$ is an element of BoundVar.

Let us consider $B$. We say that $B$ is quantifiable if and only if:

(Def. 22) There exists $e$ such that $(B_{\mathbf{1}})_{\mathbf{2}} = \mathrm{QSub}(\langle \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}}), e\rangle)$.

Let us consider $B$. Let us assume that $B$ is quantifiable. An element of vSUB is called a second q.-component of $B$ if:

(Def. 23) $(B_{\mathbf{1}})_{\mathbf{2}} = \mathrm{QSub}(\langle \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}}), \mathrm{it}\rangle)$.

In the sequel $S_4$ is a second q.-component of $B$.

Let us consider $B$, $S_4$. Let us assume that $B$ is quantifiable. The functor $\mathrm{SubAll}(B, S_4)$ yields an element of QC-Sub-WFF and is defined by:

(Def. 24) $\mathrm{SubAll}(B, S_4) = \langle \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}}), S_4\rangle$.

Let us consider $S$, $x$. Then $\langle S, x\rangle$ is an element of [: QC-Sub-WFF, BoundVar :].

The scheme *SubQCInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every element $S$ of QC-Sub-WFF holds $\mathcal{P}[S]$

provided the following conditions are satisfied:

- Let $k$ be a natural number, $P$ be a $k$-ary predicate symbol, $l_1$ be a list of variables of the length $k$, and $e$ be an element of vSUB. Then $\mathcal{P}[\mathrm{SubP}(P, l_1, e)]$,

- For every element $S$ of QC-Sub-WFF such that $S$ is sub-verum holds $\mathcal{P}[S]$,
- For every element $S$ of QC-Sub-WFF such that $\mathcal{P}[S]$ holds $\mathcal{P}[\mathrm{SubNot}(S)]$,
- For all elements $S$, $S'$ of QC-Sub-WFF such that $S_{\mathbf{2}} = S'_{\mathbf{2}}$ and $\mathcal{P}[S]$ and $\mathcal{P}[S']$ holds $\mathcal{P}[\mathrm{SubAnd}(S, S')]$, and
- Let $x$ be a bound variable, $S$ be an element of QC-Sub-WFF, and $S_4$ be a second q.-component of $\langle S, x\rangle$. If $\langle S, x\rangle$ is quantifiable and $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubAll}(\langle S, x\rangle, S_4)]$.

Let us consider $S$. We say that $S$ is sub-atomic if and only if the condition (Def. 25) is satisfied.

(Def. 25) There exists a natural number $k$ and there exists a $k$-ary predicate symbol $P$ and there exists a list of variables $l_1$ of the length $k$ and there exists an element $e$ of vSUB such that $S = \mathrm{SubP}(P, l_1, e)$.

One can prove the following proposition

(11) If $S$ is sub-atomic, then $S_{\mathbf{1}}$ is atomic.

Let $k$ be a natural number, let $P$ be a $k$-ary predicate symbol, let $l_1$ be a list of variables of the length $k$, and let $e$ be an element of vSUB. One can verify that $\mathrm{SubP}(P, l_1, e)$ is sub-atomic.

Let us consider $S$. We say that $S$ is sub-negative if and only if:

(Def. 26) There exists $S'$ such that $S = \mathrm{SubNot}(S')$.

We say that $S$ is sub-conjunctive if and only if:

(Def. 27) There exist $S_2$, $S_3$ such that $S = \mathrm{SubAnd}(S_2, S_3)$ and $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$.

Let $A$ be a set. We say that $A$ is sub-universal if and only if:

(Def. 28) There exist $B$, $S_4$ such that $A = \mathrm{SubAll}(B, S_4)$ and $B$ is quantifiable.

Next we state the proposition

(12) Every $S$ is either sub-verum, sub-atomic, sub-negative, sub-conjunctive, or sub-universal.

Let us consider $S$. Let us assume that $S$ is sub-atomic. The functor SubArguments($S$) yields a finite sequence of elements of Var and is defined by the condition (Def. 29).

(Def. 29) There exists a natural number $k$ and there exists a $k$-ary predicate symbol $P$ and there exists a list of variables $l_1$ of the length $k$ and there exists an element $e$ of vSUB such that $\mathrm{SubArguments}(S) = l_1$ and $S = \mathrm{SubP}(P, l_1, e)$.

Let us consider $S$. Let us assume that $S$ is sub-negative. The functor SubArgument($S$) yields an element of QC-Sub-WFF and is defined as follows:

(Def. 30) $S = \mathrm{SubNot}(\mathrm{SubArgument}(S))$.

Let us consider $S$. Let us assume that $S$ is sub-conjunctive. The functor SubLeftArgument($S$) yields an element of QC-Sub-WFF and is defined by:

(Def. 31) There exists $S'$ such that $S = \mathrm{SubAnd}(\mathrm{SubLeftArgument}(S), S')$ and $(\mathrm{SubLeftArgument}(S))_{\mathbf{2}} = S'_{\mathbf{2}}$.

Let us consider $S$. Let us assume that $S$ is sub-conjunctive. The functor SubRightArgument($S$) yielding an element of QC-Sub-WFF is defined as follows:

(Def. 32) There exists $S'$ such that $S = \mathrm{SubAnd}(S', \mathrm{SubRightArgument}(S))$ and $S'_{\mathbf{2}} = (\mathrm{SubRightArgument}(S))_{\mathbf{2}}$.

Let $A$ be a set. Let us assume that $A$ is sub-universal. The functor SubBound($A$) yields a bound variable and is defined as follows:

(Def. 33) There exist $B$, $S_4$ such that $A = \mathrm{SubAll}(B, S_4)$ and $B_{\mathbf{2}} = \mathrm{SubBound}(A)$ and $B$ is quantifiable.

Let $A$ be a set. Let us assume that $A$ is sub-universal. The functor SubScope($A$) yielding an element of QC-Sub-WFF is defined as follows:

(Def. 34) There exist $B$, $S_4$ such that $A = \mathrm{SubAll}(B, S_4)$ and $B_{\mathbf{1}} = \mathrm{SubScope}(A)$ and $B$ is quantifiable.

Let us consider $S$. One can verify that SubNot($S$) is sub-negative.

The following propositions are true:

(13) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{SubAnd}(S_2, S_3)$ is sub-conjunctive.

(14) If $B$ is quantifiable, then $\mathrm{SubAll}(B, S_4)$ is sub-universal.

(15) If $\mathrm{SubNot}(S) = \mathrm{SubNot}(S')$, then $S = S'$.

(16) $\mathrm{SubArgument}(\mathrm{SubNot}(S)) = S$.

(17) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$ and $(S'_1)_{\mathbf{2}} = (S'_2)_{\mathbf{2}}$ and $\mathrm{SubAnd}(S_2, S_3) = \mathrm{SubAnd}(S'_1, S'_2)$, then $S_2 = S'_1$ and $S_3 = S'_2$.

(18) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{SubLeftArgument}(\mathrm{SubAnd}(S_2, S_3)) = S_2$.

(19) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{SubRightArgument}(\mathrm{SubAnd}(S_2, S_3)) = S_3$.

(20) Let $B_1$, $B_2$ be elements of $[\!:$ QC-Sub-WFF, BoundVar $:\!]$, $S_5$ be a second q.-component of $B_1$, and $S_6$ be a second q.-component of $B_2$. If $B_1$ is quantifiable and $B_2$ is quantifiable and $\mathrm{SubAll}(B_1, S_5) = \mathrm{SubAll}(B_2, S_6)$, then $B_1 = B_2$.

(21) If $B$ is quantifiable, then $\mathrm{SubScope}(\mathrm{SubAll}(B, S_4)) = B_{\mathbf{1}}$.

The scheme *SubQCInd2* concerns a unary predicate $\mathcal{P}$, and states that:

For every element $S$ of QC-Sub-WFF holds $\mathcal{P}[S]$

provided the following requirement is met:

- Let $S$ be an element of QC-Sub-WFF. Then
  - (i) if $S$ is sub-atomic, then $\mathcal{P}[S]$,
  - (ii) if $S$ is sub-verum, then $\mathcal{P}[S]$,
  - (iii) if $S$ is sub-negative and $\mathcal{P}[\mathrm{SubArgument}(S)]$, then $\mathcal{P}[S]$,
  - (iv) if $S$ is sub-conjunctive and $\mathcal{P}[\mathrm{SubLeftArgument}(S)]$ and $\mathcal{P}[\mathrm{SubRightArgument}(S)]$, then $\mathcal{P}[S]$, and

(v) if $S$ is sub-universal and $\mathcal{P}[\mathrm{SubScope}(S)]$, then $\mathcal{P}[S]$.

One can prove the following propositions:

(22) If $S$ is sub-negative, then $\mathrm{len}(^{@}((\mathrm{SubArgument}(S))_{\mathbf{1}})) < \mathrm{len}(^{@}(S_{\mathbf{1}}))$.

(23) If $S$ is sub-conjunctive, then $\mathrm{len}(^{@}((\mathrm{SubLeftArgument}(S))_{\mathbf{1}})) < \mathrm{len}(^{@}(S_{\mathbf{1}}))$ and $\mathrm{len}(^{@}((\mathrm{SubRightArgument}(S))_{\mathbf{1}})) < \mathrm{len}(^{@}(S_{\mathbf{1}}))$.

(24) If $S$ is sub-universal, then $\mathrm{len}(^{@}((\mathrm{SubScope}(S))_{\mathbf{1}})) < \mathrm{len}(^{@}(S_{\mathbf{1}}))$.

(25)(i) If $S$ is sub-verum, then $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 0$,

(ii) if $S$ is sub-atomic, then there exists a natural number $k$ such that $(^{@}(S_{\mathbf{1}}))(1)$ is a $k$-ary predicate symbol,

(iii) if $S$ is sub-negative, then $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 1$,

(iv) if $S$ is sub-conjunctive, then $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 2$, and

(v) if $S$ is sub-universal, then $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 3$.

(26) If $S$ is sub-atomic, then $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 0$ and $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 1$ and $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 2$ and $(^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 3$.

(27) There exists no $S$ which satisfies any of the following conditions:

(i) it is sub-atomic and sub-negative,

(ii) it is sub-atomic and sub-conjunctive,

(iii) it is sub-atomic and sub-universal,

(iv) it is sub-negative and sub-conjunctive,

(v) it is sub-negative and sub-universal,

(vi) it is sub-conjunctive and sub-universal,

(vii) it is sub-verum and sub-atomic,

(viii) it is sub-verum and sub-negative,

(ix) it is sub-verum and sub-conjunctive,

(x) it is sub-verum and sub-universal.

Now we present two schemes. The scheme *SubFuncEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a function $F$ from QC-Sub-WFF into $\mathcal{A}$ such that for every element $S$ of QC-Sub-WFF and for all elements $d_1$, $d_2$ of $\mathcal{A}$ holds
>
> (i) if $S$ is sub-verum, then $F(S) = \mathcal{B}$,
>
> (ii) if $S$ is sub-atomic, then $F(S) = \mathcal{F}(S)$,
>
> (iii) if $S$ is sub-negative and $d_1 = F(\mathrm{SubArgument}(S))$, then $F(S) = \mathcal{G}(d_1)$,
>
> (iv) if $S$ is sub-conjunctive and $d_1 = F(\mathrm{SubLeftArgument}(S))$ and $d_2 = F(\mathrm{SubRightArgument}(S))$, then $F(S) = \mathcal{H}(d_1, d_2)$, and
>
> (v) if $S$ is sub-universal and $d_1 = F(\mathrm{SubScope}(S))$, then $F(S) = \mathcal{I}(S, d_1)$

for all values of the parameters.

The scheme *SubQCFuncUniq* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$ from QC-Sub-WFF into $\mathcal{A}$, a function $\mathcal{C}$ from QC-Sub-WFF into $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- Let $S$ be an element of QC-Sub-WFF and $d_1$, $d_2$ be elements of $\mathcal{A}$. Then
  (i) if $S$ is sub-verum, then $\mathcal{B}(S) = \mathcal{D}$,
  (ii) if $S$ is sub-atomic, then $\mathcal{B}(S) = \mathcal{F}(S)$,
  (iii) if $S$ is sub-negative and $d_1 = \mathcal{B}(\text{SubArgument}(S))$, then $\mathcal{B}(S) = \mathcal{G}(d_1)$,
  (iv) if $S$ is sub-conjunctive and $d_1 = \mathcal{B}(\text{SubLeftArgument}(S))$ and $d_2 = \mathcal{B}(\text{SubRightArgument}(S))$, then $\mathcal{B}(S) = \mathcal{H}(d_1, d_2)$, and
  (v) if $S$ is sub-universal and $d_1 = \mathcal{B}(\text{SubScope}(S))$, then $\mathcal{B}(S) = \mathcal{I}(S, d_1)$,
  and
- Let $S$ be an element of QC-Sub-WFF and $d_1$, $d_2$ be elements of $\mathcal{A}$. Then
  (i) if $S$ is sub-verum, then $\mathcal{C}(S) = \mathcal{D}$,
  (ii) if $S$ is sub-atomic, then $\mathcal{C}(S) = \mathcal{F}(S)$,
  (iii) if $S$ is sub-negative and $d_1 = \mathcal{C}(\text{SubArgument}(S))$, then $\mathcal{C}(S) = \mathcal{G}(d_1)$,
  (iv) if $S$ is sub-conjunctive and $d_1 = \mathcal{C}(\text{SubLeftArgument}(S))$ and $d_2 = \mathcal{C}(\text{SubRightArgument}(S))$, then $\mathcal{C}(S) = \mathcal{H}(d_1, d_2)$, and
  (v) if $S$ is sub-universal and $d_1 = \mathcal{C}(\text{SubScope}(S))$, then $\mathcal{C}(S) = \mathcal{I}(S, d_1)$.

Let us consider $S$. The functor ${}^{@}S$ yielding an element of [: WFF, vSUB :] is defined as follows:

(Def. 35) ${}^{@}S = S$.

In the sequel $Z$ denotes an element of [: WFF, vSUB :].

Let us consider $Z$. Then $Z_{\mathbf{1}}$ is an element of WFF. Then $Z_{\mathbf{2}}$ is a CQC-substitution.

Let us consider $Z$. The functor S-Bound($Z$) yields a bound variable and is defined by:

(Def. 36) $$\text{S-Bound}(Z) = \begin{cases} x_{\text{upVar}(\text{RestrictSub}(\text{Bound}(Z_{\mathbf{1}}), Z_{\mathbf{1}}, Z_{\mathbf{2}}), \text{Scope}(Z_{\mathbf{1}}))}, \\ \quad \text{if } \text{Bound}(Z_{\mathbf{1}}) \in \text{rng}\,\text{RestrictSub}(\text{Bound}(Z_{\mathbf{1}}), Z_{\mathbf{1}}, Z_{\mathbf{2}}), \\ \text{Bound}(Z_{\mathbf{1}}), \text{ otherwise.} \end{cases}$$

Let us consider $S$, $p$. The functor Quant($S, p$) yielding an element of WFF is defined by:

(Def. 37) $\mathrm{Quant}(S, p) = \forall_{\text{S-Bound}(@S)}p$.

## 3. Definition and Properties of Substitution

Let $S$ be an element of QC-Sub-WFF. The functor CQCSub($S$) yielding an element of WFF is defined by the condition (Def. 38).

(Def. 38) There exists a function $F$ from QC-Sub-WFF into WFF such that
(i) $\mathrm{CQCSub}(S) = F(S)$, and
(ii) for every element $S'$ of QC-Sub-WFF holds if $S'$ is sub-verum, then $F(S') = \mathrm{VERUM}$ and if $S'$ is sub-atomic, then $F(S') = \mathrm{PredSym}(S'_{\mathbf{1}})[\text{CQC-subst}(\mathrm{SubArguments}(S'), S'_{\mathbf{2}})]$ and if $S'$ is sub-negative, then $F(S') = \neg F(\mathrm{SubArgument}(S'))$ and if $S'$ is sub-conjunctive, then $F(S') = F(\mathrm{SubLeftArgument}(S')) \wedge F(\mathrm{SubRightArgument}(S'))$ and if $S'$ is sub-universal, then $F(S') = \mathrm{Quant}(S', F(\mathrm{SubScope}(S')))$.

We now state several propositions:

(28) If $S$ is sub-negative, then $\mathrm{CQCSub}(S) = \neg\,\mathrm{CQCSub}(\mathrm{SubArgument}(S))$.

(29) $\mathrm{CQCSub}(\mathrm{SubNot}(S)) = \neg\,\mathrm{CQCSub}(S)$.

(30) If $S$ is sub-conjunctive, then $\mathrm{CQCSub}(S) = \mathrm{CQCSub}(\mathrm{SubLeftArgument}(S)) \wedge \mathrm{CQCSub}(\mathrm{SubRightArgument}(S))$.

(31) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{CQCSub}(\mathrm{SubAnd}(S_2, S_3)) = \mathrm{CQCSub}(S_2) \wedge \mathrm{CQCSub}(S_3)$.

(32) If $S$ is sub-universal, then $\mathrm{CQCSub}(S) = \mathrm{Quant}(S, \mathrm{CQCSub}(\mathrm{SubScope}(S)))$.

The subset CQC-Sub-WFF of QC-Sub-WFF is defined by:

(Def. 39) CQC-Sub-WFF $= \{S : S_{\mathbf{1}}$ is an element of CQC-WFF$\}$.

Let us observe that CQC-Sub-WFF is non empty.

Next we state several propositions:

(33) If $S$ is sub-verum, then CQCSub($S$) is an element of CQC-WFF.

(34) Let $h$ be a finite sequence. Then $h$ is a variables list of $k$ if and only if $h$ is a finite sequence of elements of BoundVar and $\mathrm{len}\, h = k$.

(35) CQCSub(SubP($P, l_1, e$)) is an element of CQC-WFF.

(36) If CQCSub($S$) is an element of CQC-WFF, then CQCSub(SubNot($S$)) is an element of CQC-WFF.

(37) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$ and CQCSub($S_2$) is an element of CQC-WFF and CQCSub($S_3$) is an element of CQC-WFF, then CQCSub(SubAnd($S_2, S_3$)) is an element of CQC-WFF.

In the sequel $x_1$ denotes a second q.-component of $\langle S, x \rangle$.

We now state the proposition

(38) If CQCSub($S$) is an element of CQC-WFF and $\langle S, x \rangle$ is quantifiable, then CQCSub(SubAll($\langle S, x \rangle, x_1$)) is an element of CQC-WFF.

In the sequel $S$ is an element of CQC-Sub-WFF.

The scheme *SubCQCInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every $S$ holds $\mathcal{P}[S]$

provided the following requirement is met:

- Let $S$, $S'$ be elements of CQC-Sub-WFF, $x$ be a bound variable, $S_4$ be a second q.-component of $\langle S, x \rangle$, $k$ be a natural number, $l_1$ be a variables list of $k$, $P$ be a $k$-ary predicate symbol, and $e$ be an element of vSUB. Then
  - (i) $\mathcal{P}[\mathrm{SubP}(P, l_1, e)]$,
  - (ii) if $S$ is sub-verum, then $\mathcal{P}[S]$,
  - (iii) if $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubNot}(S)]$,
  - (iv) if $S_{\mathbf{2}} = S'_{\mathbf{2}}$ and $\mathcal{P}[S]$ and $\mathcal{P}[S']$, then $\mathcal{P}[\mathrm{SubAnd}(S, S')]$, and
  - (v) if $\langle S, x \rangle$ is quantifiable and $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubAll}(\langle S, x \rangle, S_4)]$.

Let us consider $S$. Then $\mathrm{CQCSub}(S)$ is an element of CQC-WFF.

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