Benedikt Löwe, Wolfgang Malzkorn, Thoralf Räsch
Foundations of the Formal Sciences II
Applications of Mathematical Logic in Philosophy and Linguistics
Bonn, November 10-13, 2000, pp. 1–**??**.

A Refinement of Jensen's Constructible Hierarchy

Peter Koepke and Marc van Eijmeren[⋆]
Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
Beringstraße 1
D-53115 Bonn
Germany
E-mail: koepke@math.uni-bonn.de, mveijmer@math.uni-bonn.de

Abstract. Appropriate hierarchies are essential for the study of minimal or constructible inner models of set theory, in particular for proving strong combinatorial principles in those models. We define a finestructural hierarchy (F_α) and compare it to Jensen's well-known J_α-hierarchy for Gödel's constructible universe L.

1 Introduction

Under the axioms of Zermelo-Fraenkel set theory (ZF) the universe V of all sets can be layered into a hierarchy of initial segments which is indexed by the ordinal numbers. John von Neumann made the following recursive definition: $\mathrm{V}_0 := \emptyset$, $\mathrm{V}_{\alpha+1} := \mathcal{P}(\mathrm{V}_\alpha)$, and $\mathrm{V}_\lambda := \bigcup_{\alpha<\lambda} \mathrm{V}_\alpha$, for limit ordinals λ. The axiom schema of foundation (among others) implies that this hierarchy exhausts the set theoretical universe: $\mathrm{V} := \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{V}_\alpha$. A set x is of ordinal *rank* α if $x \in \mathrm{V}_{\alpha+1} \setminus \mathrm{V}_\alpha$. The rank function measures the complexity of x in terms of the membership relation.

Received: ...;
In revised version: ...;
Accepted by the editors: ....
2000 *Mathematics Subject Classification.* **PRIMARY** SECONDARY.

⋆ The second author was funded by the Mathematical Institute of the University of Bonn and the DAAD.

Transfinite induction along the ordinals is essentially the only method to carry out involved arguments and constructions on infinite sets. Via the von Neumann hierarchy, some inductive arguments can be applied to all sets in the universe V.

In his fundamental work on the relative consistency of the axiom of choice and of the generalised continuum hypothesis, Kurt Gödel defined the *constructible universe* L which is the *smallest* inner model of ZF. The model L has become the prototype for a canonical model of set theory. The constructible universe is defined and analysed through definability hierarchies.

The present paper illustrates that there exists a variety of appropriate hierarchies for L. We briefly recall the two classical examples: the L_α-hierarchy by Gödel and the J_α-hierarchy by Jensen. We shall then introduce a new hierarchy (F_α) for L and relate it to Jensen's hierarchy. The main technical result will be that the Jensen-hierarchy consists of the limit levels of the F_α-hierarchy which underlines again the canonical nature of the Jensen-hierarchy.

Gödel [Göd39] defined the hierarchy $(\mathrm{L}_\alpha)_{\alpha\in\mathrm{Ord}}$ by iterating a *definable powerset* operation instead of the unrestricted powerset operation. Denoting the collection of all first-order definable subsets of $(X, \in)$ by $\mathrm{Def}(X)$, the hierarchy is defined by: $\mathrm{L}_0 := \emptyset$, $\mathrm{L}_{\alpha+1} := \mathrm{Def}(\mathrm{L}_\alpha)$, and $\mathrm{L}_\lambda := \bigcup_{\alpha<\lambda} \mathrm{L}_\alpha$, for limit ordinals λ. The sets formed in the various levels of this hierarchy are called the *constructible sets*, $\mathrm{L} := \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{L}_\alpha$ is defined to be the *constructible universe*.

The elements in $\mathrm{L}_{\alpha+1} = \mathrm{Def}(\mathrm{L}_\alpha)$ can be classified according to their defining formulas over $(\mathrm{L}_\alpha, \in)$ and to the parameters used. The formulas can be indexed by natural numbers, and the parameters have already been located in some earlier level. This allows to denote every set in L by a finite sets of ordinals or, indeed, by one ordinal so that the methods of induction and recursion can be applied to all elements of L. In this way, Gödel proved that the class L is a model of set theory, the *inner model* of constructible sets, in which the axiom of choice and the generalized continuum hypothesis hold.

The constructible universe satisfies many combinatorial properties and allows involved constructions that cannot be carried out under the ZFC axioms alone. Ronald Jensen analysed the process of set formation in the constructible universe in his paper *The fine structure of the constructible hierarchy* [Jen72] for which he received the *2003 AMS Steele*

Prize for a Seminal Contribution to Research. The theory is based on the analysis of first-order formulas which define a set in $\mathrm{L}_{\alpha+1} \setminus \mathrm{L}_\alpha$. Jensen's approach requires a thorough syntactical study of formulas of arbitrary quantifier complexity.

For his work, Jensen introduced the J-hierarchy for L, whose levels are closed with respect to *rudimentary* functions. These are the functions generated by the scheme:

- constant functions and projection functions are rudimentary;
- the formation of unordered pairs is rudimentary;
- if $f(x_0, \ldots, x_{n-1})$ and $g_0(\vec{y}), \ldots, g_{n-1}(\vec{y})$ are rudimentary functions then the composition $f(g_0(\vec{y}), \ldots, g_{n-1}(\vec{y}))$ is rudimentary;
- if g is a rudimentary function then $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$ is rudimentary.

The smallest, rudimentary closed set containing X is denoted by $\mathrm{rud}(X)$. Then the J-hierarchy is defined recursively as follows: $\mathrm{J}_0 := \emptyset$, $\mathrm{J}_{\alpha+1} := \mathrm{rud}(\mathrm{J}_\alpha \cup \{\mathrm{J}_\alpha\})$, and $\mathrm{J}_\lambda := \bigcup_{\alpha<\lambda} \mathrm{J}_\alpha$ for limit ordinals λ.

The finestructure theory augments the structures J_α with a host of additional components like canonical well-orders, Skolem functions and truth predicates in order to reduce arbitrary definability over some J_α to Σ_1-definability over the augmented structure. The fundamental lemmas of the theory ensure that the additional components are preserved under various operations in the hierarchy. We state some properties of the J-hierarchy. Proofs can be found in [Dev84, Chap. VI].

Lemma 1. There exists a $\mathbf{\Sigma}_1(\mathrm{J}_\alpha)$ map from $\omega\alpha$ onto $\omega\alpha \times \omega\alpha$.

Lemma 2. There exists a $\mathbf{\Sigma}_1(\mathrm{J}_\alpha)$ map from $\omega\alpha$ onto J_α.

Theorem 1. *For every $\alpha \geq 1$, $\mathcal{P}(\mathrm{J}_\alpha) \cap \mathrm{J}_{\alpha+1} = \mathbf{\Sigma}_\omega(\mathrm{J}_\alpha)$. So in particular $\mathbf{\Sigma}_\omega(\mathrm{J}_\alpha) \subseteq \mathrm{J}_{\alpha+1}$ for every $\alpha \geq 1$.*

Recently, some variants of finestructure theory have been proposed which try to keep the "model theoretic" intuitions of Jensen's theory whilst reducing the syntactical complexities. S. D. Friedman and the first author [FrKo97] e.g. modified the standard L_α-hierarchy in a way which allows to use the combinatorics of Silver machines [Sil] in combinatorial proofs whilst retaining Gödel's iterated definability approach.

In this note, we present yet another fine structural hierarchy (F_α). The central idea is to incorporate canonical well-orders and Skolem functions directly into the hierarchy, so that these do not have to be *defined* over the structures. The F-hierarchy is defined via *quantifier-free definability*, which simplifies fine structural arguments. It has been used by the first author to provide a simple proof of the Jensen covering theorem. Details and proofs will be published in [Koe∞]. In the present article we prove that the F-hierarchy is a refinement of the J-hierarchy; the J-hierarchy consists of the limit structures of the F-hierarchy, hence the F-hierarchy is an adequate hierarchy for the constructible universe. The proof uses the fact that sufficiently much of the recursion for the F-hierarchy can be carried out inside the structures J_α.

The original motivation for introducing the F-hierarchy was *core model theory*. The F-hierarchy can be used nicely for structuring the Dodd-Jensen core model K, but there are problems with higher core models so far.

2 The F-hierarchy

We introduce the F-hierarchy and state some basic properties. We approximate the constructible universe by a hierarchy $(\mathrm{F}_\alpha)_{\alpha \in \mathrm{Ord}}$ of structures $\mathrm{F}_\alpha = (\mathrm{F}_\alpha, I_\alpha, <_\alpha, S_\alpha, \in)$. Each F_α will be a transitive set and $\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{F}_\alpha = \mathrm{L}$. The functions I_α and S_α are the restrictions to $\mathrm{F}_\alpha^{<\omega}$ of a global *interpretation function* I and a global *Skolem function* S defined on $\mathrm{L}^{<\omega}$. The relation $<_\alpha$ is the restriction of a ternary *guarded constructible well-order* $<$ to F_α^3. Let us first define a "Skolem" language $\mathcal{S}$ adequate for the structures F_α:

Definition 1. Let $\mathcal{S}$ be the first-order language with the following components:

- variable symbols $\dot{v}_n$ for $n < \omega$;
- logical symbols $\dot{=}$ (equality), $\dot{\wedge}$ (conjunction), $\dot{\neg}$ (negation), $\dot{\exists}$ (existential quantification), (,) (brackets);
- function symbols $\dot{I}$ (interpretation), $\dot{S}$ (Skolem function) of variable finite arity;
- a binary relation symbol $\dot{\in}$ (set-membership) and a *ternary* relation symbol $\dot{\prec}$ (guarded well-order).

The syntax and semantics of $\mathcal{S}$ are defined as usual. The variable arity of $\dot{I}$ and $\dot{S}$ is handled by bracketing: if $n < \omega$ and $t_0, \ldots, t_{n-1}$ are

$\mathcal{S}$-terms then $\dot{I}(t_0,\ldots,t_{n-1})$ and $\dot{S}(t_0,\ldots,t_{n-1})$ are $\mathcal{S}$-terms. If t_0, t_1, t_2 are $\mathcal{S}$-terms then $t_0 \dot{\in} t_1$ and $t_0 \dot{\prec}_{t_1} t_2$ are atomic $\mathcal{S}$-formulas. We also denote the set of first-order $\mathcal{S}$-formulas by $\mathcal{S}$.

We assume that $\mathcal{S}$ is Gödelized in an effective way: the set of formulas satisfies $\mathcal{S} \subseteq \omega^{<\omega}$, and the usual syntactical operations of $\mathcal{S}$ are recursively definable over V_ω. This includes the simultaneous substitution $\varphi\frac{\vec{t}}{\vec{w}}$ of terms $\vec{t}$ for variables $\vec{w}$ in φ. The notation $\varphi(\dot{v}_0,\ldots,\dot{v}_{n-1})$ indicates that the free variables of φ are contained in $\{\dot{v}_0,\ldots,\dot{v}_{n-1}\}$. By $\mathcal{S}_0$ we denote the collection of *quantifier-free* formulas of $\mathcal{S}$. The formulas of $\mathcal{S}$ are interpreted in $\mathcal{S}$-structures in the obvious way. If $\mathcal{A} = (A, \dot{I}^{\mathcal{A}}, \dot{\prec}^{\mathcal{A}}, \dot{S}^{\mathcal{A}}, \dot{\in}^{\mathcal{A}})$ is an $\mathcal{S}$-structure, $\varphi(\dot{v}_0,\ldots,\dot{v}_{n-1}) \in \mathcal{S}$ and $a_0,\ldots,a_{n-1} \in A$ then $\mathcal{A} \models \varphi[a_0,\ldots,a_{n-1}]$ says that $\mathcal{A}$ is a model of φ under the variable assignment $\dot{v}_i \mapsto a_i$ for $i < n$.

The F_α-hierarchy is defined by iterated $\mathcal{S}_0$-definability:

Definition 2. The *fine hierarchy* consists of a monotone sequence of $\mathcal{S}$-structures

$$\mathrm{F}_\alpha = (\mathrm{F}_\alpha, I, \prec, S, \in) = (\mathrm{F}_\alpha, I \restriction \mathrm{F}_\alpha, \prec\restriction \mathrm{F}_\alpha, S \restriction \mathrm{F}_\alpha, \in)$$

where $\mathrm{F}_\alpha, I \restriction \mathrm{F}_\alpha, \prec\restriction \mathrm{F}_\alpha, S \restriction \mathrm{F}_\alpha$ are defined by recursion on $\alpha \in \mathrm{Ord}$: For $\alpha \leq \omega$ let $\mathrm{F}_\alpha = \mathrm{V}_\alpha$ and $\forall \vec{x} \in \mathrm{F}_\alpha : I(\vec{x}) = S(\vec{x}) = 0$. Let $<_\omega$ be a recursive binary relation which well-orders $\mathrm{F}_\omega = \mathrm{V}_\omega$ in ordertype ω and which extends the $\in$-relation on $\{\mathrm{V}_n \mid n < \omega\}$. Define the ternary relation $\prec$ on F_α by: $x \prec_z y$ iff ($z = \mathrm{F}_n$ for some $n < \alpha$, $x, y \in z$ and $x <_\omega y$). This defines the structures F_α for $\alpha \leq \omega$.

Assume that $\alpha \geq \omega$ and that $\mathrm{F}_\alpha = (\mathrm{F}_\alpha, I, \prec, S, \in)$ has been defined. For $\varphi(\dot{v}_0,\ldots,\dot{v}_n) \in \mathcal{S}_0$ and $\vec{p} \in \mathrm{F}_\alpha$ set

$$I(\mathrm{F}_\alpha, \varphi, \vec{p}) = \{x \in \mathrm{F}_\alpha \mid \mathrm{F}_\alpha \models \varphi[x, \vec{p}]\}. \tag{*}$$

In this case, $(\mathrm{F}_\alpha, \varphi, \vec{p})$ is a *name* for its *interpretation* $I(\mathrm{F}_\alpha, \varphi, \vec{p})$ in the fine hierarchy. The next fine level is defined as

$$\mathrm{F}_{\alpha+1} = \{I(\mathrm{F}_\alpha, \varphi, \vec{p}) \mid \varphi \in \mathcal{S}_0, \vec{p} \in \mathrm{F}_\alpha^{<\omega}\}.$$

We have to define $I(\vec{z})$ for "new" vectors $\vec{z} \in (\mathrm{F}_{\alpha+1})^{<\omega} \setminus \mathrm{F}_\alpha^{<\omega}$. Certain assignments were made in (*); in all other cases set $I(\vec{z}) = 0$.

Define the ternary relation $\prec\restriction \mathrm{F}_{\alpha+1}$ extending $\prec\restriction \mathrm{F}_\alpha$ by adjoining the following triples: $x \prec_{\mathrm{F}_\alpha} y$ iff $x, y \in \mathrm{F}_\alpha$ and there is a name $(\mathrm{F}_\beta, \varphi, \vec{p})$

for x such that every name $(\mathrm{F}_\gamma, \psi, \vec{q})$ for y is lexicographically greater than $(\mathrm{F}_\beta, \varphi, \vec{p})$, where coordinates are well-ordered from left to right by $\in$, $<_\omega$ and $\prec_{\mathrm{F}_{\max(\beta,\gamma)}}$ respectively.

The Skolem function S finds witnesses of existential statements. We only need to define $S(\vec{z})$ for $\vec{z} \in (\mathrm{F}_{\alpha+1})^{<\omega} \setminus \mathrm{F}_\alpha^{<\omega}$. Set $S(\vec{z}) = 0$ except when $\vec{z} = (\mathrm{F}_\alpha, \varphi, \vec{p})$, where $\varphi \in \mathcal{S}_0$, $\vec{p} \in \mathrm{F}_\alpha^{<\omega}$ and $I(\mathrm{F}_\alpha, \varphi, \vec{p}) \neq \emptyset$. In that case let $S(\vec{z})$ be the $\prec_{\mathrm{F}_\alpha}$-least element of $I(\mathrm{F}_\alpha, \varphi, \vec{p})$.

This defines $\mathrm{F}_{\alpha+1} = (\mathrm{F}_{\alpha+1}, I, \prec, S, \in)$.

Assume that $\lambda > \omega$ is a limit ordinal and that F_α is defined for $\alpha < \lambda$. Then the limit structure $\mathrm{F}_\lambda = (\mathrm{F}_\lambda, I, \prec, S, \in)$ is defined by unions: $\mathrm{F}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{F}_\alpha$, $I \upharpoonright \mathrm{F}_\lambda = \bigcup_{\alpha<\lambda} I \upharpoonright \mathrm{F}_\alpha$, $\prec \upharpoonright \mathrm{F}_\lambda = \bigcup_{\alpha<\lambda} \prec \upharpoonright \mathrm{F}_\alpha$, $S \upharpoonright \mathrm{F}_\lambda = \bigcup_{\alpha<\lambda} S \upharpoonright \mathrm{F}_\alpha$.

The fine hierarchy satisfies some natural hierarchical properties some of which were already tacitly assumed in the previous definition:

Proposition 1. For every $\gamma \in \mathrm{Ord}$:

1. $\alpha \leq \gamma \rightarrow \mathrm{F}_\alpha \subseteq \mathrm{F}_\gamma$;
2. $\alpha < \gamma \rightarrow \mathrm{F}_\alpha \in \mathrm{F}_\gamma$;
3. F_γ is transitive.

First-order definability can be emulated in $\mathcal{S}_0$:

Proposition 2. For every $\in$-formula $\varphi(\dot{v}_0, \ldots, \dot{v}_{n-1})$ one can uniformly define a quantifier free formula $\varphi^*(\dot{v}_0, \ldots, \dot{v}_{n-1}, \dot{v}_n, \ldots, \dot{v}_{n+k}) \in \mathcal{S}_0$ such that for all $\alpha \geq \omega$ and for all $a_0, \ldots, a_m \in \mathrm{F}_\alpha$:

$$(\mathrm{F}_\alpha, \epsilon) \models \varphi[a_0, \ldots, a_{n-1}] \iff \mathrm{F}_{\alpha+k} \models \varphi^*[a_0, \ldots, a_{n-1}, \mathrm{F}_\alpha, \mathrm{F}_{\alpha+1}, \ldots, \mathrm{F}_{\alpha+k}].$$

Proposition 3. Every $\mathrm{F}_{\omega\alpha}$ is closed with respect to first order definability.

Theorem 2. $\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{F}_\alpha = \mathrm{L}$.

So we have defined a hierarchy for the constructible universe. Proofs of the above theorem and propositions can be found in [Koe∞].

We show that the F_α-hierarchy is a refinement of the J_α-hierarchy:

Theorem 3. *For every* $\alpha \in \mathrm{Ord}$, $\mathrm{F}_{\omega\alpha} = \mathrm{J}_\alpha$.

Proof. The proof of this result occupies the rest of this paper and consists of a sequence of lemmas. Using Proposition 3 one can easily prove one inclusion.

Lemma 3. For every $\alpha \in \mathrm{Ord}$, $\mathrm{J}_\alpha \subseteq \mathrm{F}_{\omega\alpha}$.

Proof. By induction on α. For $\alpha = 0$ and $\alpha = 1$ this is true by definition, so assume $\mathrm{J}_\gamma \subseteq \mathrm{F}_{\omega\gamma}$ for every $\gamma < \alpha$. J_α is the rudimentary closure of the J_γ's. As rudimentary functions are first-order definable, $\mathrm{F}_{\omega\alpha}$ is rudimentarily closed by Proposition 3. Hence $\mathrm{F}_{\omega\alpha}$ includes J_α.
qed (Lemma 3).

For the other inclusion, we show that the F-hierarchy is absolute for each J_α where $\alpha > 1$, i.e., the above recursive definition of the F-hierarchy can be carried out within J_α and will define the same sequence as the recursive definition in V for all ordinals in J_α. Since J_α is rudimentary closed and contains the language $\mathcal{S}$ as an element, we see by inspection that the recursive conditions in the definition of the F-hierarchy are absolute for J_α. Properties like

$$I(\mathrm{F}_\gamma, \varphi, \vec{p}) = \{x \in \mathrm{F}_\gamma \mid \mathrm{F}_\gamma \models \varphi[x, \vec{p}]\}$$

or

$$\mathrm{F}_{\gamma+1} = \{I(\mathrm{F}_\gamma, \varphi, \vec{p}) \mid \varphi \in \mathcal{S}_0, \vec{p} \in \mathrm{F}_\gamma^{<\omega}\}$$

refer to a quantifier-free satisfaction relation. Evaluating this within J_α involves evaluating finite sequences of values of terms and truth-values according to the structure of the formula φ considered. These finite sequences exist in the structure J_α just like they exist in the universe V. The other recursive conditions in Definition 2 are absolute for similar reasons. So the recursion can indeed be carried out absolutely in J_α provided we can prove the following
Claim. All ordinals α satisfy $\forall \mu < \omega\alpha\ (\mathrm{F}_\nu \mid \nu < \mu) \in \mathrm{J}_\alpha$.
This can be shown by induction on α. The **limit case** is trivial: if α is a limit ordinal the inductive hypothesis implies that for all $\beta < \alpha$, all initial segments of the F-hierarchy of length $< \omega\beta$ are in J_β. This yields the property for J_α.

Now consider the **successor case**: assume that $\alpha = \beta + 1$ and that $\forall \mu < \omega\beta\ (\mathrm{F}_\nu \mid \nu < \mu) \in \mathrm{J}_\beta$. Then $(\mathrm{F}_\nu \mid \nu < \omega\beta)$ is definable over J_β by the recursion formula and is thus an element of J_α. To show that the sequences $(\mathrm{F}_\nu \mid \nu < \mu)$ for $\omega\beta < \mu < \omega\alpha$ are elements of J_α it suffices to show by induction on ν, $\omega\beta \leq \nu < \omega\alpha$ that $\mathrm{F}_\nu \in \mathrm{J}_\alpha$.

Since the claim holds for β, $(\mathrm{F}_\nu \mid \nu < \omega\beta)$ is recursively definable over J_β. Then the structure $\mathrm{F}_{\omega\beta}$ is definable over J_β as the union of that tower of structures. Thus $\mathrm{F}_{\omega\beta} \in \mathrm{J}_\beta$.

For the successor step of this inner induction, we use a surjection $h \in \mathrm{J}_{\beta+1}$ from $\omega\beta$ onto $\mathrm{F}_{\omega\beta+n}$ to define an $\mathrm{F}_{\omega\beta+n}$-like structure over $\omega\beta$, hence over J_β. We can then define the next level of the F-hierarchy within $\mathrm{J}_{\beta+1}$. We start with some technical lemmas.

Lemma 4. Let $x \in \mathrm{J}_{\beta+1}$ and let $h \in \mathrm{J}_{\beta+1}$ be a surjection from $\omega\beta$ onto x. Then $x^{<\omega} \in \mathrm{J}_{\beta+1}$.

Proof. The set $(\omega\beta)^{<\omega} = \{s \mid \mathrm{Func}(s) \wedge \mathrm{dom}(s) \in \omega \wedge \mathrm{ran}(s) \subseteq \omega\beta\}$ is a definable subset of J_β. Hence $(\omega\beta)^{<\omega} \in \mathrm{J}_{\beta+1}$. Since $\mathrm{J}_{\beta+1}$ is rudimentarily closed:

$$\begin{aligned} x^{<\omega} &= \{h \circ s \mid s \in (\omega\beta)^{<\omega}\} \\ &= \bigcup_{s \in (\omega\beta)^{<\omega}} h \circ s \\ &\in \mathrm{J}_{\beta+1}. \end{aligned}$$

qed (Lemma 4).

For a surjection h from $\omega\beta$ onto x we denote by $\vec{h}$ the surjection from $(\omega\beta)^{<\omega}$ onto $x^{<\omega}$ induced by h: for any $\vec{s} = (\xi_0, \ldots, \xi_{n-1})$ set

$$\vec{h}(\vec{s}) := h \circ \vec{s} = (h(\xi_0), \ldots, h(\xi_{n-1})).$$

Then $h \in \mathrm{J}_{\beta+1}$ implies that $\vec{h} \in \mathrm{J}_{\beta+1}$.

Lemma 5. For every $\alpha > 1$, $\mathcal{S}_0 \in \mathrm{J}_\alpha$.

Proof. In [Dev84, VI, 1.14], a language similar to $\mathcal{S}$ is defined over J_1. Therefore, $\mathcal{S}_0$ is definable over J_1 and an element of J_α for $\alpha > 1$.

qed (Lemma 5).

The following lemma then concludes the proof of our theorem:

Lemma 6. Let $\nu = \omega\beta + n$, $\mathrm{F}_\nu = (\mathrm{F}_\nu, I, \prec, S, \in) \in \mathrm{J}_{\beta+1}$, h_ν a surjection from $\omega\beta$ onto F_ν and $h_\nu \in \mathrm{J}_{\beta+1}$. Then $\mathrm{F}_{\nu+1} \in \mathrm{J}_{\beta+1}$ and there is a surjection $h_{\nu+1}$ from $\omega\beta$ onto $\mathrm{F}_{\nu+1}$ such that $h_{\nu+1} \in \mathrm{J}_{\beta+1}$.

Proof. Define a structure $\bar{\mathrm{F}}_\nu = (\bar{\mathrm{F}}_\nu, i_\nu, \prec_\nu, s_\nu, \epsilon_\nu)$ over $\omega\beta$ which is analogous to F_ν:

$$\begin{aligned}
\bar{\mathrm{F}}_\nu &:= \{\gamma \in \omega\beta \mid h_\nu(\gamma) \in \mathrm{F}_\nu\} \\
i_\nu(\varphi, \vec{s}) &:= \{\gamma \in \omega\beta \mid h_\nu(\gamma) \in I_\nu(\varphi, \vec{h}_\nu(\vec{s}))\} \\
\prec_\nu &:= \{(\gamma, \delta) \mid (h_\nu(\gamma), h_\nu(\delta)) \in <_\nu\} \\
\epsilon_\nu &:= \{(\gamma, \delta) \mid h_\nu(\gamma) \in h_\nu(\delta)\}
\end{aligned}$$

These are all subsets of J_β and elements of $\mathrm{J}_{\beta+1}$, hence definable over J_β. We complete the structure by defining:

$$\begin{aligned}
i_\nu &:= \bigcup_{\substack{\varphi \in \mathcal{S}_0 \\ \vec{s} \in (\omega\beta)^{<\omega}}} \{((\varphi, \vec{s}), i_\nu(\varphi, \vec{s}))\} \\
s_\nu &:= \bigcup_{\vec{s} \in (\omega\beta)^{<\omega}} \{(\vec{s}, s_\nu(\vec{h}_\nu(\vec{s})))\} \\
\mathbf{f}_\nu &:= (f_\nu, i_\nu, \prec_\nu, s_\nu, \epsilon_\nu).
\end{aligned}$$

The latter definitions are compositions of rudimentary functions and elements of $\mathrm{J}_{\beta+1}$, hence $\mathbf{f}_\nu \in \mathrm{J}_{\beta+1}$.

We can now extend $\mathbf{f}_\nu$ to a structure $\mathbf{f}_{\nu+1} = (f_{\nu+1}, i_{\nu+1}, \prec_{\nu+1}, s_{\nu+1})$ in a similar way as in the F-hierarchy:

$$
\begin{aligned}
i_{\nu+1}(\varphi,\vec{s}) &:= \{\gamma \in \omega\beta \mid \mathbf{f}_\nu \models \varphi[\gamma,\vec{s}]\}\\
i_{\nu+1} &:= \bigcup_{\substack{\varphi\in\mathcal{S}_0\\ \vec{s}\in(\omega\beta)^{<\omega}}} \{((\varphi,\vec{s}), i_{\nu+1}(\varphi,\vec{s}))\}\\
f_{\nu+1} &:= \bigcup_{\substack{\varphi\in\mathcal{S}_0\\ \vec{s}\in(\omega\beta)^{<\omega}}} \{i_{\nu+1}(\varphi,\vec{s})\}\\
\prec_{\nu+1} &:= \prec_\nu \cup \{(\gamma,\delta) \mid \gamma \in f_\nu \wedge \delta \in f_{\nu+1} \setminus f_\nu\}\\
&\quad \cup \{(\gamma,\delta) \mid \exists\varphi,\vec{s}\,\forall\psi,\vec{t}(\gamma = i_{\nu+1}(\varphi,\vec{s})\\
&\qquad \wedge \delta = i_{\nu+1}(\psi,\vec{t}) \wedge (\varphi \in \psi \vee (\varphi = \psi \wedge \vec{s} <_{\text{lex}} \vec{t})))\\
s_{\nu+1}(\vec{s}) &:= \begin{cases} s_\nu(\vec{s}) & \text{if } \vec{s} \in \operatorname{dom}(s_\nu),\\ \text{the } \prec_\nu\text{-least of } i_{\nu+1}(\varphi,\vec{t}) & \text{if } \vec{s} \notin \operatorname{dom}(s_\nu) \text{ and}\\ & \vec{s} \text{ is of the shape } (\nu,\varphi,\vec{t}),\\ \emptyset & \text{otherwise.}\end{cases}\\
s_{\nu+1} &:= \bigcup_{\vec{s}\in(\omega\beta)^{<\omega}} \{(\vec{s}, s_{\nu+1}(\vec{s}))\}
\end{aligned}
$$

We look closer at the definitions to show that the structure $\mathbf{f}_{\nu+1}$ is an element of $\mathrm{J}_{\beta+1}$. First we remark that similarly to ordinary Σ_0-satisfaction, the $\mathbf{f}$-satisfaction relation for quantifier-free formulae is rudimentary. For ordinary Σ_0-satisfaction this carried out in detail in [Dev84]. Therefore $i_{\nu+1} \in \mathrm{J}_{\beta+1}$ and $f_{\nu+1} \in \mathrm{J}_{\beta+1}$.

To see that $\prec_{\nu+1} \in \mathrm{J}_{\beta+1}$ we note first that the first part of the definition is an element of $\mathrm{J}_{\beta+1}$ by assumption and the second as well as we know $f_{\nu+1} \in \mathrm{J}_{\beta+1}$. For the third part we note that

$$
\begin{aligned}
&\vec{s} <_{\text{lex}} \vec{t} \iff\\
&\qquad \vec{s} \neq \vec{t} \wedge (\operatorname{dom}(\vec{s}) < \operatorname{dom}(\vec{t}) \vee (\operatorname{dom}(\vec{s}) = \operatorname{dom}(\vec{t}) \wedge\\
&\qquad ((\forall n \in \operatorname{dom}(\vec{s}))\, \vec{s}(n) > \vec{t}(n) \rightarrow (\exists m < n)\, s(m) \prec_\beta t(m))))
\end{aligned}
$$

Finally the Skolem function $s_{\nu+1}$ is readily definable from $\prec_\nu$ and $i_{\nu+1}$, hence is an element of $\mathrm{J}_{\beta+1}$. So $\mathbf{f}_{\nu+1} := (f_{\nu+1}, i_{\nu+1}, \prec_{\nu+1}, s_{\nu+1}) \in \mathrm{J}_{\beta+1}$.

We now define

$$\begin{aligned}
\mathrm{F}_{\nu+1} &:= \bigcup_{\gamma \in f_{\nu+1}} h_\nu(\gamma) \\
i'_{\nu+1}(\varphi, \vec{s}) &:= \{h_\nu(\gamma) \mid \gamma \in i_{\nu+1}\} \\
I_{\nu+1} &:= \bigcup_{\substack{\varphi \in \mathcal{S}_0 \\ \vec{s} \in (\omega\beta)^{<\omega}}} \{((\varphi, \vec{h}_\nu(\vec{s})), i'_{\nu+1}(\varphi, \vec{s}))\} \\
<_{\nu+1} &:= \{(h_\nu(\gamma), h_\nu(\delta) \mid (\gamma, \delta) \in \prec_{\nu+1}\} \\
S_{\nu+1} &:= \{(\vec{h}_\nu(\vec{s}), h_\nu(\gamma)) \mid (\vec{s}, \gamma) \in s_{\nu+1}\} \\
\mathrm{F}_{\nu+1} &:= (\mathrm{F}_{\nu+1}, I_{\nu+1}, <_{\nu+1}, S_{\nu+1})
\end{aligned}$$

By a now familiar argument we find $\mathrm{F}_{\nu+1} \in \mathrm{J}_{\beta+1}$

We still need to construct a surjection $h_{\nu+1}$ from $\omega\beta$ onto $F_{\nu+1}$. We can define a surjection g from $\omega\beta$ onto $\omega^{<\omega} \times (\omega\beta)^{<\omega}$ (see for example [Dod82]). Let g' be its restriction to $\mathcal{S}_0 \times (\omega\beta)^{<\omega}$. Then we define $h_{\nu+1} := h_\nu \circ i_{\nu+1} \circ g'$. It is easily seen this is a surjection and by construction we have $g' \in \mathrm{J}_{\beta+1}$, hence $h_{\nu+1} \in \mathrm{J}_{\beta+1}$. qed (Lemma 6).

In all, we have proved Theorem 3. **QED.**

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