Einstein's Clocks

How can identical clocks measure time at different rates?
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Einstein's theory of Special Relativity started with thought experiments that analyzed the concept of simultaneity. It took 50 years before more and more experiments started to be performed that verified Einstein's predictions to higher and higher accuracy. 25 years ago relativity entered daily life when the global positioning system (GPS) was built. But many people still react with complete disbelief to the statement that time may pass with different speed – nowadays measurably in many situations. I will first try to explain what the precise meaning of this statement is and then show that this basic fact of relativity theory is in perfect agreement with other facts from physics that are less difficult to accept.

What are clocks?

The first precision time pieces were pendulum clocks. They had one imperfection that caused problems in astronomy and made them useless on the ocean: When transported they lost their precision completely. Time pieces with balance springs were much better behaved and quartz clocks essentially did not loose precision when transported. These classical clocks have a common principle: They have a very regular but delicate clock pulse generator, a mechanism that counts the pulses, translates the count into and shows the time that passed, and finally an energy source that keeps the pulse generator going. Presently our standard time is measured with atomic clocks. Basically the clock pulse generator is the transition frequency between two energy levels of the element cesium and the point is that the transition radiation has an extraordinarily narrow band width. More technically, a microwave radiation of approximately this transition frequency is synthesized and its absorption by the cesium atoms is used to regulate it to precisely the correct transition frequency. Again, the frequency is counted and the count is translated into time that passed. Let me now emphasize that it does not really matter what opinions about time one has. But one needs to realize that all statements involving time in physics mean the time that is **measured**, presently by cesium clocks. For example, the physicists and the engineers involved in the installation of the global positioning system did not agree about how time passes. Therefore two different counting systems had to be installed in the early GPS satellites. The non-relativistic version was so far off that the system did not work.

The choice of the element cesium for our standard clocks has technical reasons for achieving high precision. In principle one can use the transition frequency between any two energy levels of any atom. I connect this fact with a fundamental astronomical observation: If one observes spectral lines in the light of any celestial object, then one can identify subfamilies of lines as the lines of specific elements **because the ratios of the celestial lines are the same as the ratios in our laboratories!** Rephrased as time measurements this says: The atomic clocks at any place in the universe (that we have been able to observe) agree among each other about how time passes at that place. The fact that the ratios agree and not the frequencies themselves means that we are observing clocks which agree among each other but their time passes with different speed than ours. We will see that relative motion, the so called Doppler effect, can explain this.

Observation of identical clocks that tick with different speed.

What kind of an experiment could one imagine that lets us observe identical clocks ticking with different speed? In principal we could sit next to one clock and observe another one ticking differently. A skeptic would still blame the clock rather than accept our statement about time passing differently. Therefore I want to describe another type of clock that, admittedly, cannot be built to the precision of a cesium clock, but they convey such a robust impression of the passing of time that I find it very difficult to disbelieve them. These clocks measure the passing of time with radioactive material: one unit of time of such a clock has passed, if one half of the original amount of material has decayed. Several such clocks are in practical use in archaeology. There is no indication so far that they might not agree with the cesium clocks. Now, if we hand to two physicists equal blocks of radium, let them go their ways and when they later meet again we count the radium atoms they have left. If one of them has 5% fewer than the other aren't we forced to say that for him 5% more time has passed? – Well, except for the skeptical remark: I would prefer to see such an experiment instead of speculating about its possible outcome.

Already when I was a student the physicists Pound and Rebka performed such an experiment. They put one (generalized) atomic clock on the ground floor of a 40 m tower and an identical clock at the top. The bottom clock sent its time signals to the top. Technically simpler, the bottom clock sent directly the transition radiation of its clock pulse generator to the top clock. The newly discovered Mösbauer effect had to be used so that the emitted radiation did not loose momentum to the emitting atom. At the top Pound and Rebka observed that the incoming frequency was too slow to be absorbed by the identical atoms of the top clock pulse generator. In other words: they observed that the bottom clock was ticking more slowly than the top clock! Even more surprising, they could determine how much too slow the bottom clock was and found that the difference in clock speed was in perfect agreement with older well established facts from physics. Here are the details:
From an electromagnetic wave one can absorb energy only in portions

$$E = h \cdot \nu$$

These portions are called photons.
These photons have a mass m

$$E = m \cdot c^2,\ m = E/c^2 = h \cdot \nu/c^2$$

Here c denotes the speed of light.
If some mass m flies a distance s upwards in the gravitational field of the earth, then it looses the following amount of kinetic energy

$$\Delta E = m \cdot g \cdot s.$$

Pound and Rebka found that also photons loose exactly this much energy! We can translate this energy change into a frequency change:

$$\Delta\nu = \frac{\Delta E}{h} = \nu \cdot \frac{g \cdot s}{c^2},$$

and it is exactly this frequency change that was responsible for the bottom clock to tick more slowly. (Summary on transparency 1 at the end.)

> In other words, if we accept the two Nobel prize formulas above and the Pound Rebka measurement (made possible by the Mösbauer Nobel prized effect) then the time signals of the bottom clock arrive at the top clock at the slowed down rate predicted by energy conservation!

Clocks in motion relative to each other.

Now we turn to the origin of Special Relativity. Decades before cesium clocks and the Pound Rebka experiment Einstein predicted on the basis of thought experiments that relative motion would affect how identical clocks measure time. History shows that many people are unable to accept Einstein's analysis (among them were even the engineers of the GPS project). I believe one reason is that we have absolutely no every day experience with observations made by two people in fairly fast motion relative to each other. Therefore I chose the Pound Rebka experiment as introduction: An observer can sit quietly and watch the two identical clocks tick at different rate. This situation is so simple that one cannot argue with its set up.

In Einstein's 1905 analysis there was no gravity. We are asked to imagine two observing physicists in whose laboratories one cannot measure the faintest traces of any acceleration. However they are allowed to be in constant relative motion. As far as we know, the laws of physics have to be exactly the same in all such situations. This is now postulated as **the principle of relativity** and neither experiments nor theoretical analysis raise any suspicion that this principle might be wrong. Such laboratories are called inertial systems. Note that such inertial systems are an idealization which does not exist in our world. Einstein's falling elevator can only turn off a strictly homogenous gravitational field, not the real fields we live in. Therefore no practical reference frame will be strictly inertial. Special Relativity is part of the ideal world of inertial systems and its acceptance has to be in this idealized form. Its assumptions are never strictly satisfied, in no real or imagined laboratory of our world.

Let me recall that the situation is the same with our 3-dimensional Euclidean geometry. We are completely at ease in using this ideal geometry, although we can never check whether our physical surroundings strictly satisfy its axioms. Let me recall one property of Euclidean geometry which is very similar to what we will meet in Special Relativity. We are accustomed to use coordinates called Height, Width and Depth, they measure distances in three orthogonal directions. Given these orthogonal measurements we compute the length ℓ of a vector (x, y, z) with the Pythagorean theorem as $\ell = \sqrt{x^2 + y^2 + z^2}$. Then we discover that this formula is not tied to our standard coordinates: we can take any three pairwise

orthogonal unit vectors $\{e_1, e_2, e_3\}$, write $(x, y, z) = x_1e_1 + x_2e_2 + x_3e_3$ and find the surprising fact that the length is always computed by the same formula: $\ell = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In other words, although we usually think of having a naturally preferred coordinate system it is true that all the other coordinates are equally good and no geometric difference between them exists. In a completely analogous way we will describe the geometry of Special Relativity first from the point of view of one preferred inertial observer and then we discover that in all inertial systems the same formulas hold. The analogy goes still farther. Of course we know from our Euclidean geometry the following: If we join two distinct points in space by two different curves then we find it silly to expect that the two curves have the same length. If we accept the geometry of Special Relativity with the same trust then the famous twin paradox goes away by turning silly: the time measured by a clock is the length of that curve that describes the traveling life of the clock, and length means *length with respect to the geometry of Special Relativity.* As in the Euclidean analogue: it is silly to expect that different curves have the same length.
To derive the geometry of Special Relativity we only use the **principle of relativity** and a fundamental hypothesis formulated by Einstein: *The traveling speed of a light signal is independent of the motion of its source*, or in more colloquial words: the **speed of light is constant**. Physicists had met this constant traveling speed of electromagnetic waves already before Einstein, in Maxwell's theory of electromagnetism. And briefly before Einstein published 'Special Relativity', further support was given to the constant speed of light hypothesis by the (negative) result of the Michelson-Morley interferometer experiment.

The Geometry of Special Relativity.

What we have to understand can be condensed into the following main problem. Consider two inertial observers whose inertial systems have the velocity v relative to each other. We assume further that they meet at some moment and set their clocks to zero at that instant. When their clocks show time 1 each of them sends a light signal towards the other one (moving away with velocity v).

> The time T when these light signals are received will be the same for both of these inertial observers because of the relativity principle.

How large is T?

To answer this question they agree to return a light signal at the moment when the first signal is received (i.e. at clock time T). The first signals were sent at clock time 1 and received at clock time T. For the second pair of signals the time intervals are stretched by a factor T: sent at clock time T and received at clock Time T^2. Both of them use the same clocks, hence the same units of time. To measure lengths they agree on units such that the speed of light is $c = 1$. Now both of them can plot the world line of the other and the world lines of the light signals in coordinates of their inertial systems, see transparency 2 at the end. Our observers solve two linear equations and find T^2, hence T, in terms of c and v.

This fundamental relation gives the factor T by which the time between two received light signals is longer than the time between the emissions of these signals (positive v means moving apart). This frequency shift is called the **Doppler effect**.

$$T^2 = \frac{c+v}{c-v}.$$

Now both can mark on the world line of the other(!) the points where the clock time is one. They now observe that for **all inertial observers** the Time-1-Points on the other world lines satisfy (in their own coordinates) the equation of a two-sheeted hyperboloid:

$$t^2 - |x|^2 = 1.$$

The two physicists therefore have achieved for Special Relativity what corresponds, in our Euclidean 3-space, to the determination of the unit ball.
The quadratic expression $t^2 - |x|^2$ plays for Special Relativity the same role that the Pythagorean theorem plays for Euclidean space. In particular it determines the time-like arc length on world lines without reference to any(!) observer. But this time-like arc length on a world line is the time that an atomic clock having this world line does measure: *Measured time is a geometric property of the world line in question.*

In the last statement we apply the insight that we obtained for inertial observers more generally to accelerated observers, in other words: to curved world lines. We justify this generalization by noting that the time-like arc length of a curved world line can be obtained by approximating the world line by piecewise straight, i.e. non-accelerated, world lines. Since the corners of such approximations are not physically meaningful one might also want to see experimental support. Indeed, we can observe particles with a very short lifetime circling at high speed in a synchroton. Not only do we notice immediately that they circle many more times than their lifetime permits, we also find after doing the computation (see transparency 3 at the end) that the number of completed orbits is exactly what the computed passing of time on these world lines allows them. Notice that this is a twin paradox experiment: a twin particle watching from the center of the synchroton its orbiting twin will reach the end of its life time long before the orbiting particle decays. Put differently, it is not difficult to imagine two physicists starting their rather different lives with two equal chunks of radium. When they meet again late in life it would be a colossal coincidence if the time-like arc lengths of their world lines really were the same. Therefore they will find their remaining chunks of radium to be of different size.

One can even observe the different passing of time in (fairly) inertial systems and on inertial world lines. Of course, in such a situation the two world lines cannot have the same start point and the same end point. For a full explanation it would therefore be necessary to discuss how distances are measured in the two inertial systems. This requires more definitions than just clocks. Therefore we only mention the experiment without detailed explanation:

Collisions by cosmic rays generate high in the atmosphere very short lived but fast traveling mesons. They are measured in a laboratory about 30 kilometers away. Even with the speed of light they could not travel 30 km in their life time. However the Time-1-Point for their world line is given by Minkowski's hyperbola and the result is that much less proper time passes on the meson's world line from the top of the atmosphere to the ground laboratory than passes on the world line of a rocket that flies between the same places. Therefore its life time suffices to reach the ground.

Summary and repetition:
1.) Since, according to Pound and Rebka, photons flying upwards in a gravitational field loose the same amount of (kinetic) energy as a mass $m = h\nu/c^2$ gains in potential energy, the frequency ν of the corresponding wave is decreased by the same percentage. This can be rephrased by saying: the distance between time signals increases by the same percentage. Therefore we watch the clock that is higher up in the gravitational field ticking faster by exactly this percentage.
2.) The principle of relativity and the constancy of the speed of light imply that the Time-1-Points on unaccelerated world lines in inertial systems lie on the gauge surface $t^2 - |x|^2 = 1$. This allows to define a time-like arc length on world lines and our analysis of clocks means that this time-like arc length is the (so called proper) time that passes along such a world line and is measured by atomic clocks or decaying radium.

Experiments that support Special Relativity:
http://www.atomki.hu/fizmind/specrel/experiments.html
Clock debate before the start of GPS satellites:
http://www.leapsecond.com/history/Ashby-Relativity.htm

Transparency 1

1.) According to Max Planck one can absorb energy from an electromagnetic wave only in portions

$$E = h \cdot \nu.$$

These portions are called photons.

2.) These photons have a mass m **according to Einstein's famous formula**

in general: $E = m \cdot c^2,$ **for photons:** $m = \frac{h \cdot \nu}{c^2},$

where c **denotes the speed of light.**

3.) If some mass m **flies a height** s **upwards in the gravitational field of the earth, then it looses the following amount of kinetic energy**

$$\Delta E = m \cdot g \cdot s.$$

4.) The experiment of Pound and Rebka shows that the same is true for photons

$$\Delta E = \frac{h \cdot \nu}{c^2} \cdot g \cdot s.$$

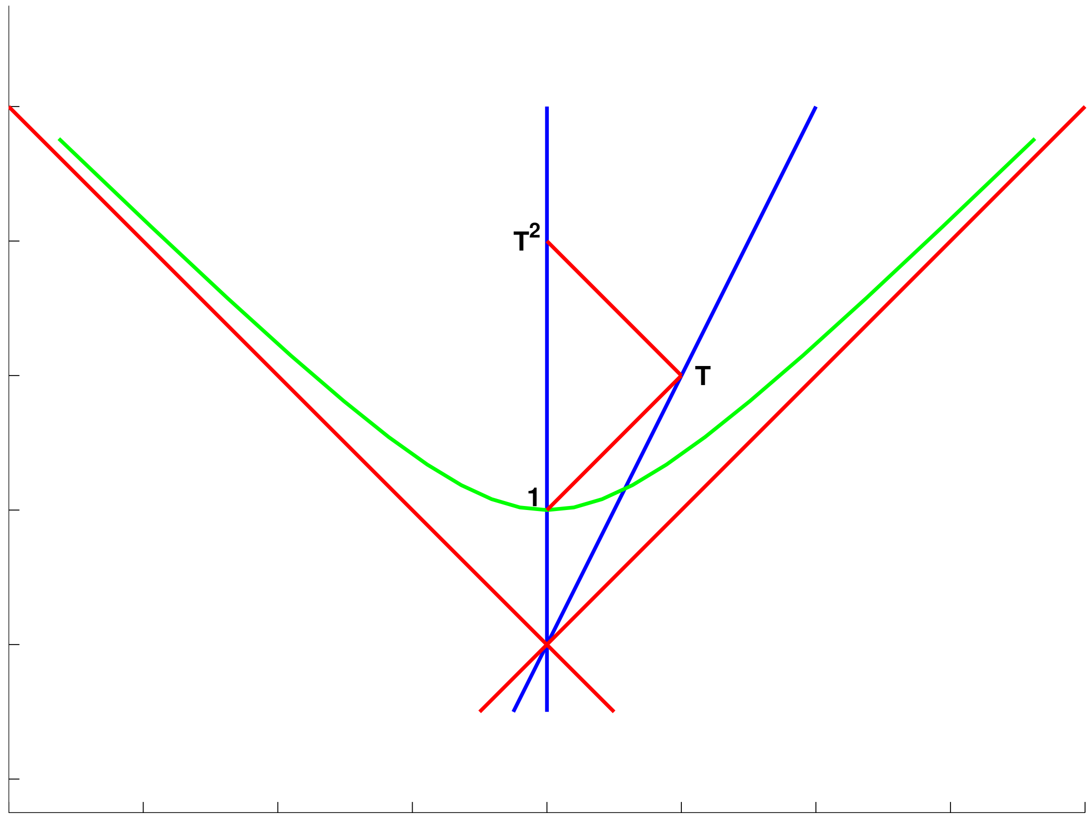
This energy change translates into a frequency change

$$\Delta\nu = \frac{\Delta E}{h} = \nu \cdot \frac{g \cdot s}{c^2}.$$

5.) A clock which is a height s **above another clock in the field of the earth ticks faster by this same percentage**

$$\frac{\Delta\nu}{\nu} = \frac{g \cdot s}{c^2}!$$

The Time 1 Points of Minkowski Geometry



1.) World line of a light signal starting from 1 (red): $\binom{0}{1} + \binom{1}{1} \cdot t$
The world line of the second observer, starting at 0 (blue): $\binom{a}{1} \cdot s$
The intersection of these two world lines (at yet unknown clock time T) is:

$$\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} \cdot \frac{a}{1-a} = \begin{pmatrix}a\\1\end{pmatrix} \cdot \frac{1}{1-a}.$$

2.) The returning signal is received at clock time T^2 in

$$\begin{pmatrix}0\\T^2\end{pmatrix} = \begin{pmatrix}0\\\frac{1+a}{1-a}\end{pmatrix}, \textbf{hence } T = \sqrt{\frac{1+a}{1-a}}.$$

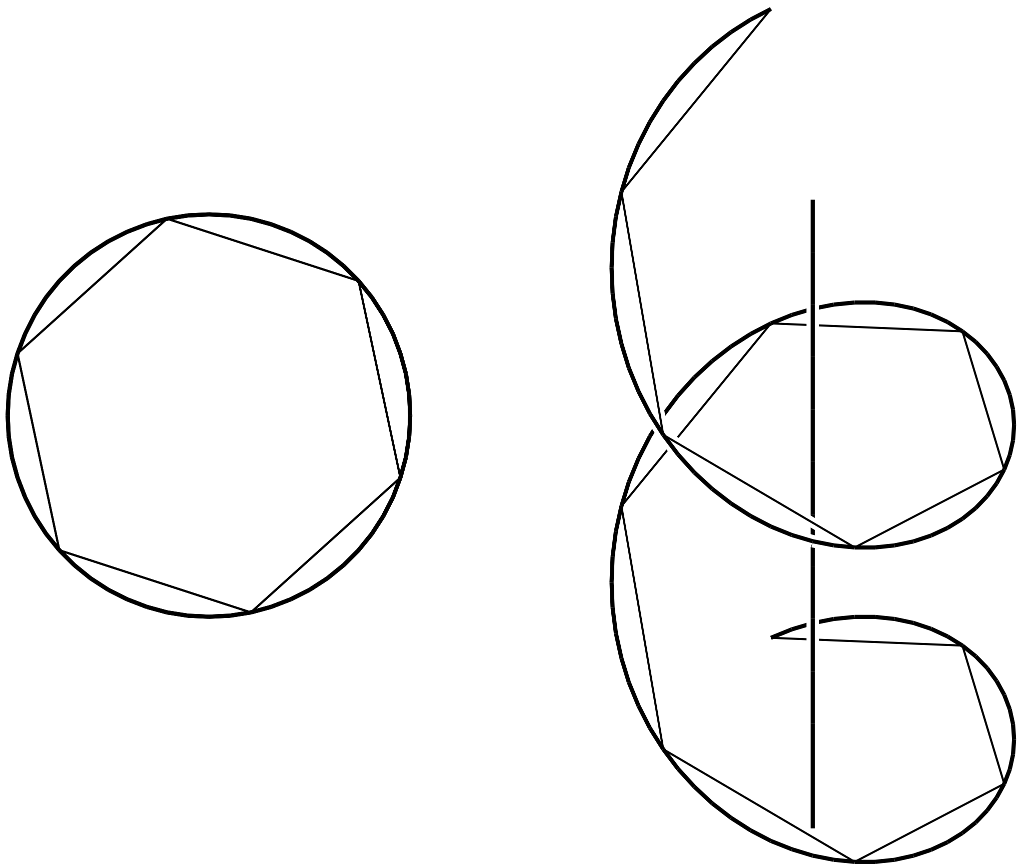
3.) The Time-1-Point on the second world line therefore is at

$$\frac{1}{T} \cdot \begin{pmatrix}a\\1\end{pmatrix} \cdot \frac{1}{1-a} = \frac{1}{\sqrt{1-a^2}} \cdot \begin{pmatrix}a\\1\end{pmatrix}.$$

4.) All the Time-1-Points $\binom{x}{t}$ (green) therefore satisfy the following hyperbola equation

$$t^2 - x^2 = 1.$$

How Time passes in a Synchroton



It is a theorem that the arc length of smooth curves in Euclidean space can be determined via approximation by polygons. The same proof shows that the time-like arc length of world lines can be determined via approximation by piecewise non-accelerated worldlines, even though the corners of these approximations are physically unrealistic. Since we have found the Time-1-Points on straight world lines we can conclude how time passes on the world lines of particles circling in the synchroton. Such a world line is a

$$\textbf{helix: } c(s) := \begin{pmatrix} \cos(s) \\ \sin(s) \\ h \cdot s \end{pmatrix}, \qquad h > 1$$

and time passes as

$$T(s) = \sqrt{h^2 - 1} \cdot s,$$

while on the world line that is the axis of the helix the larger time $T_{axis} = h \cdot s$ **passes.**

It is a correct idea to imagine time as time-like arc length of world lines.