OVERCONVERGENT HILBERT MODULAR FORMS VIA PERFECTOID MODULAR VARIETIES

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Abstract. We give a new construction of p-adic overconvergent Hilbert modular forms by using Scholze's perfectoid Shimura varieties at infinite level and the Hodge–Tate period map. The definition is analytic, closely resembling that of complex Hilbert modular forms as holomorphic functions satisfying a transformation property under congruence subgroups. As a special case, we first revisit the case of elliptic modular forms, extending recent work of Chojecki, Hansen and Johansson. We then construct sheaves of geometric Hilbert modular forms, as well as subsheaves of integral modular forms, and vary our definitions in p-adic families. We show that the resulting spaces are isomorphic as Hecke modules to earlier constructions of Andreatta, Iovita and Pilloni. Finally, we give a new direct construction of sheaves of arithmetic Hilbert modular forms, and compare this to the construction via descent from the geometric case.

1. Introduction

In a first introduction, modular forms are usually defined as certain holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ on the complex upper half-plane $\mathcal{H}$ satisfying a transformation property of the form

$$\gamma^* f = (cz + d)^k f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

where $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup. More algebraically, one can consider modular forms as sections of an automorphic line bundle ω on the complex modular curve $\Gamma\backslash\mathcal{H}$. This algebraic definition admits a p-adic interpretation, giving rise to the theory of overconvergent modular forms varying in p-adic families, which has proved extremely important with wide-ranging applications in algebraic number theory and arithmetic geometry.

An analytic definition of p-adic overconvergent modular forms has, however, proved elusive, until such an approach was recently introduced in the case of rational quaternionic modular forms by Chojecki, Hansen and Johansson [CHJ17].

In this article, we give an analytic definitions of both arithmetic and geometric p-adic Hilbert modular forms over any totally real field F, for any prime p, and show that they agree with earlier algebraic definitions of Andreatta–Iovita–Pilloni in [AIP16a].

Following [CHJ17], the key idea of the construction is to use Scholze's perfectoid Shimura varieties at infinite level over a perfectoid base field L, and the associated Hodge–Tate period map π_{HT}, all introduced in [Sch15]. These spaces can be viewed as p-adic analogues of $\mathcal{H}$. In the complex situation, the pullback of the automorphic bundle ω along the covering map

$$\mathcal{H} \to \Gamma\backslash\mathcal{H}$$

can be canonically trivialised, and the descent to $\Gamma\backslash\mathcal{H}$ via the action of Γ gives rise to the usual definition of complex modular forms, at least after dealing with compactifications.

Similarly, in the p-adic situation, there is an adic analytic moduli space $\mathcal{X}$, which in our case is a Hilbert modular variety, carrying an automorphic bundle ω. It has a cover

$$\mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{X}$$

by a perfectoid Hilbert moduli space. Using π_{HT}, the pullback of ω along this projection can be canonically trivialised over open subspaces. Via the action of the associated covering group – a p-adic level subgroup – one obtains a definition of overconvergent Hilbert modular forms.

More precisely, there are two different kinds of Hilbert modular forms: there are those associated to the group $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, which are called *arithmetic*; and those associated to $G^* := G \times_{\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m} \mathbb{G}_m$, which are called *geometric*. Shimura varieties for G^* have a moduli

interpretation in terms of abelian varieties with PEL structure. For any (narrow) ideal class $\mathfrak{c} \in \mathrm{Cl}^+(F)$, we consider the $\mathfrak{c}$-polarised finite level Shimura variety $\mathcal{X}_{\mathfrak{c}}$. Since the Shimura variety for G^* is in particular of Hodge type, one gets an associated infinite level Hilbert moduli space $\mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}$ in the limit over level structures at p. Inside of this we have for any small enough $\epsilon \geq 0$ an open subspace $\mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a$, the ϵ-overconvergent anticanonical locus.

Definition 1.1. For any weight character $\kappa : \mathbb{Z}_p^\times \to L^\times$, a *geometric overconvergent Hilbert modular form of weight* κ is a function $f \in \mathcal{O}(\mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a)$ satisfying

$$\gamma^* f = \kappa^{-1}(c\mathfrak{z} + d) f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^*(p), \tag{1.1}$$

where $\Gamma_0^*(p)$ is a p-adic level subgroup, and $\kappa(c\mathfrak{z} + d)$ is a factor of automorphy to be defined.

From modular forms for G^*, one can obtain an indirect definition of modular forms for G by descent. As we shall show, one of the advantages of the analytic approach is that instead, one can also work with perfectoid Shimura varieties attached to G, and give a completely intrinsic definition. Let $\mathcal{X}_{G,\mathfrak{c}}$ be the $\mathfrak{c}$-polarised Shimura variety for G. This is now not a fine moduli space of abelian varieties, but one can still construct a perfectoid cover

$$\mathcal{X}_{G,\mathfrak{c},\Gamma(p^\infty)} \to \mathcal{X}_{G,\mathfrak{c}}.$$

Definition 1.2. An *arithmetic overconvergent Hilbert modular form of weight* κ is a function $f \in \mathcal{O}(\mathcal{X}_{G,\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a)$ satisfying

$$\gamma^* f = \kappa^{-1}(c\mathfrak{z} + d) w_\kappa(\det \gamma) f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{P}\Gamma_0(p)$$

where $\mathrm{P}\Gamma_0(p)$ is a p-adic level subgroup, and w_κ is a character to be defined.

More generally, one can similarly define line bundles ω^κ whose global sections are the modular forms of Definitions 1.1 and 1.2. These bundles, and hence the modular forms, vary naturally over p-adic families $\mathcal{U}$ in the respective weight spaces, by considering analytic functions on the sousperfectoid space $\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a := \mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \times_L \mathcal{U}$ (and analogously for G).

1.1. **What is new.** Several constructions of both geometric and arithmetic p-adic overconvergent Hilbert modular forms have already appeared in the literature (e.g. amongst others [KL05], [AIP16a] and most generally [AIP16b]), so let us explain how our constructions differ and what, in our opinion, are some of their advantages.

- Our main goal is to give a new intrinsic definition of the sheaf of arithmetic Hilbert modular forms for G, which is arguably cleaner and easier to work with than the one via descent from G^*.
- We also show how to define subspaces of integral geometric and arithmetic Hilbert modular forms in the analytic setting, which match up with the ones constructed in [AIP16b]. An advantage of the perfectoid construction is that this does not require formal models. Rather, the subspace of integral forms is given by simply replacing $\mathcal{O}$ with the integral subsheaf $\mathcal{O}^+$ in the construction.
- As in [CHJ17], the resulting framework is well-adapted to constructing overconvergent Eichler–Shimura maps from overconvergent cohomology, namely maps of the form

$$H_c^g(\mathcal{X}_{\mathfrak{c}}, D_\kappa) \to H^0(\mathcal{X}_{\mathfrak{c}}^{\mathrm{tor}}(\epsilon), \omega^\kappa \otimes \Omega^g_{\mathcal{X}_{\mathfrak{c}}^{\mathrm{tor}}(\epsilon)}(-\partial))(-g)$$

  where D_κ is an étale sheaf of distribution modules, $g = [F : \mathbb{Q}]$, and $\partial \subseteq \mathcal{X}_{\mathfrak{c}}^{\mathrm{tor}}(\epsilon)$ is the boundary of a chosen toroidal compactification. A proof of this will be included in upcoming work.

As a secondary goal, we modify the strategy of [CHJ17] in several ways:

- We work with the anticanonical locus rather than the canonical one, which makes it easier to deal with the boundary of the Shimura varieties, an issue which is is not present in *op.cit.* as there the construction is carried out for quaternionic modular forms.

  As a minor but pleasant side effect, this results in the automorphic factor $\kappa(c\mathfrak{z} + d)$ appearing in (1.1), like in the complex case, rather than the $\kappa(b\mathfrak{z} + d)$ from *op.cit.*
- We give a conceptually new proof that the resulting sheaves are line bundles.

- Throughout we work with sousperfectoid spaces, a language that was not available at the time that [CHJ17] was written. This allows one to define automorphic sheaves uniformly for arbitrary bounded weights $\mathcal{U}$ in a geometric way, by working over the fibre product of the infinite level modular variety with $\mathcal{U}$. In particular, one does not have to impose restrictions on the shape of weights as in the construction using formal models.
- We also explain how the "perfect" modular forms of [AIP16b] appear in this anticanonical setting. In the elliptic case, this point of view has been used in [Heu19] for a perfectoid approach to Coleman's Spectral Halo, and one should be able to use our constructions to obtain similar results for Hilbert modular forms.

We shall now explain our constructions and the organisation of the paper in some more detail.

1.2. **Elliptic modular forms via the anticanonical locus.** While the main focus of this paper is to construct families of Hilbert modular forms, we start in §2, 3 and 4, by treating the elliptic case. One reason to consider this seperately is that while the boundary in the higher dimensional case can be dealt with via Koecher's principle, in the elliptic case it requires an explicit analysis. Our second reason to treat the elliptic case separately is to illustrate the ways in which we deviate from the construction in [CHJ17].

To explain this, we first summarise their construction. Let L be any perfectoid field over $\mathbb{Q}_p^{\mathrm{cyc}}$, let $\mathcal{X}^*$ be the (compact) modular curve over L of some tame level considered as an adic space, and let $q : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{X}^*$ be Scholze's infinite level perfectoid modular curve (denoted $\mathcal{X}_\infty$ *op. cit.*). It admits a Hodge–Tate period map $\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1$ with the key property

$$\pi_{\mathrm{HT}}^*\mathcal{O}(1) = q^*\omega.$$

To study this sheaf, they consider a family of open subspaces of $\mathbb{P}^1$, parametrised by $w \in \mathbb{Q}_{>0}$, on which $\mathcal{O}(1)$ admits a non-vanishing section. Pulling back under π_{HT} gives a family of neighbourhoods $\mathcal{X}^*_{\Gamma(p^\infty),w} \subset \mathcal{X}^*_{\Gamma(p^\infty)}$ of the (canonical) ordinary locus. There are then subspaces $\mathcal{X}^*_w \subseteq \mathcal{X}^*$ such that $\mathcal{X}^*_{\Gamma(p^\infty),w} \to \mathcal{X}^*_w$ is a pro-étale $\Gamma_0(p)$-torsor, at least away from the cusps. Here $\Gamma_0(p) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ is the subgroup of matrices that are upper-triangular modulo p.

Pulling back the natural parameter at $\infty \in \mathbb{P}^1$, they obtain a parameter $\mathfrak{z} \in \mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty),w})$. For a certain class of p-adic weights κ, and $w \geq 0$ sufficiently small, they then define the space of "w-overconvergent" modular forms of weight κ to be the set of $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty),w})$ satisfying

$$\gamma^* f = \kappa(b\mathfrak{z} + d)^{-1} f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p).$$

1.2.1. *The case of elliptic modular forms.* The results of [CHJ17] are only explicitly proved in the quaternionic case where the Shimura curve is compact, though they do mention that their methods can be extended to the elliptic case, where one can use "soft" techniques to deal with ramification at the boundary. This is also noted in [How17]. We expand on these remarks and explain how to extend to the cusps by using perfectoid Tate curve parameter spaces.

Instead of considering the canonical locus, we choose to work with the anticanonical locus $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ everywhere. This definition is equivalent, since the two loci can be interchanged via the action of the Atkin–Lehner matrix $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$.We define a sheaf

$$\omega_1^\kappa := \left\{ f \in q_*\mathcal{O}_{\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a} \middle| \gamma^* f = \kappa^{-1}(c\mathfrak{z} + d) f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p) \right\}$$

on $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$, where now $\mathfrak{z}$ is the parameter on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ defined by pulling back the canonical parameter on $\mathbb{A}^{1,\mathrm{an}} \subseteq \mathbb{P}^1$ at 0, and $q : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{X}^*_{\Gamma_0(p)}$ is the projection. The space of ϵ-overconvergent modular forms is then the space of global sections of this sheaf. We note that this is very similar to the complex definition. We then use the Atkin–Lehner isomorphism to obtain a sheaf $\omega^\kappa = \omega_0^\kappa := \mathrm{AL}^*\omega_1^\kappa$ on the tame level space $\mathcal{X}^*(p^{-1}\epsilon)$. One could now prove, as in [CHJ17], that the sheaf ω^κ is a line bundle, but we instead deduce this from our later comparison results.

1.2.2. *Variation in families.* Reinterpreting [CHJ17] in the context of sousperfectoid spaces, we show how to extend the definition to also work for p-adic families. The weights considered above can be considered as points $\mathrm{Spa}(L, \mathcal{O}_L) \xrightarrow{\kappa} \mathcal{W}$ in the rigid analytic *weight space*

$$\mathcal{W} := \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\mathrm{ad}}_\eta \times L,$$

where throughout we consider all rigid spaces as adic spaces, and where L is our perfectoid field. We can then consider more general (families of) weights $\kappa : \mathcal{U} \to \mathcal{W}$, where $\mathcal{U}$ is a smooth rigid space and κ has bounded image. This gives rise to a sheaf ω^κ on the fibre product $\mathcal{X}^*(p^{-1}\epsilon) \times_L \mathcal{U}$ whose fibre over any point $\kappa_0 \in \mathcal{U}$ is the sheaf ω^{κ_0} defined above.

By comparing the anticanonical locus with the Pilloni-torsor as described in [AIP18], in Thm. 4.8 we construct an isomorphism between ω^κ and the bundle $\omega^\kappa_{\mathrm{AIP}}$ of forms *op. cit.*:

Theorem 1.3. *Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight (Defn. 3.1; e.g. a point or an affinoid open in $\mathcal{W}$). Then there is a natural Hecke-equivariant isomorphism $\omega^\kappa \cong \omega^\kappa_{\mathrm{AIP}}$.*

One reason we prefer to work with the anticanonical locus over the canonical one is that it simplifies the proof of the above comparison. A second reason is that it makes it easier to study the boundary: the technical complication for defining elliptic modular forms rather than quaternionic ones is that the cover $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ is pro-étale over $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$, but is ramified at the cusps. However, the situation at the cusps is easy to deal with in the anticanonical tower, because here the cusps are totally ramified and give rise to perfectoid versions of Tate curve parameter discs. This allows one to extend the arguments from [CHJ17] to the boundary.

1.3. **Generalisation to the Hilbert case.** The main result of the present paper is a generalisation of this approach to the setting of Hilbert modular forms, that is, modular forms for GL_2 over any totally real field F of degree g.

Having treated the elliptic case separately, we will assume $g \geq 2$, and, by the Koecher principle, largely ignore the boundary in this case. Whilst conceptually the constructions follow the same lines as in the elliptic case, there are additional subtleties in the Hilbert case that do not arise when the base field is $\mathbb{Q}$. The most immediate is in the choice of *classical* definition. The Shimura varieties arising from G^* are of PEL (hence Hodge) type. They are fine moduli spaces parametrising Hilbert–Blumenthal abelian varieties (HBAVs), namely abelian varieties equppied with an $\mathcal{O}_F$-action and a polarisation, plus some fixed tame level structure. The Shimura varieties for G, in contrast, are only of abelian type, and are only coarse moduli spaces, parametrising instead only equivalence classes of polarisations. These distinctions make it technically easier to work with G^*, although ultimately the case in which we are most interested is the arithmetic case of G, which has a better theory of Hecke operators.

In both cases these Shimura varieties are called Hilbert modular varieties, for G^* and G respectively. In practice, we will work with the $\mathfrak{c}$-polarised part of the Hilbert modular variety. We sometimes emphasize this with a subscript $\mathfrak{c}$, but usually drop this from the notation.

1.3.1. *Hilbert modular varieties for G^* at infinite level.* As in the elliptic case, the key object in the construction of overconvergent forms is an infinite level Hilbert modular variety for G^*, which is a p-adic analogue of the classical complex Hilbert modular variety. It it is a special case of Scholze's perfectoid Shimura varieties of Hodge type [Sch15, §III and IV]. As we shall recall in §5.2, it arises from the tower of ($\mathfrak{c}$-polarised) Hilbert modular varieties $\mathcal{X}_{\mathfrak{c},\Gamma^*(p^n)}$ as the wild level $\Gamma^*(p^n) \subset G^*(\mathbb{Z}_p)$ varies. Once again, one can restrict to the anticanonical locus of an ϵ-neighbourhood of the ordinary locus and obtain a Hilbert modular variety at infinite level

$$\mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\mathfrak{c},\Gamma^*(p^n)}(\epsilon)_a,$$

which is a pro-étale $\Gamma_0^*(p)$-torsor over $\mathcal{X}_{\mathfrak{c},\Gamma_0^*(p)}(\epsilon)_a$, where $\Gamma_0^*(p) \subset G^*(\mathbb{Z}_p)$ is the subgroup of matrices that are upper-triangular modulo p.

We also need a version of the Hodge–Tate period map, as defined in [Sch15, §IV] and refined in [CS17, §2]. If C is a perfectoid field extension of L, then a (C, C^+)-point of $\mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}$ corresponds to a HBAV A equipped with a trivialisation $\alpha : \mathcal{O}_p^2 \xrightarrow{\sim} T_p A^\vee$ and extra data, where $\mathcal{O}_p := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Here we note that the appearance of $A^\vee$ differs somewhat from [Sch15], but in the presence of a polarisation λ, the two are always isomorphic after a choice of p-adic generator of $\mathfrak{c}$. The reason we wish to parametrise $T_p A^\vee$ rather than $T_p A$ is that together with the Hodge–Tate morphism

$$\mathcal{O}_p^2 \xrightarrow{\alpha} T_p A^\vee \xrightarrow{\mathrm{HT}_A} \omega_A,$$

the trivialisation α gives rise to canonical differentials $\alpha(1,0)$ and $\alpha(0,1)$ of ω_A (instead of $\omega_{A^\vee}$).

This construction can be made more conceptual by way of the Hodge–Tate period morphism

$$\pi_{\mathrm{HT}} : \mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \longrightarrow \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\mathbb{P}^1.$$

If F splits in L (which we do not assume in the main text), this decomposes canonically into maps $\pi_{\mathrm{HT}} = \prod_{v\in\Sigma}\pi_{\mathrm{HT},v} : \mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \to (\mathbb{P}^1)^\Sigma$, where Σ is the set of embeddings $v : F \hookrightarrow L$. On C-points, a point corresponding to an isomorphism α is then sent to the point in $(\mathbb{P}^1(C))^\Sigma$ defined by the Hodge filtration

$$0 \to \mathrm{Lie}(A^\vee)(1) \longrightarrow T_pA^\vee \otimes_{\mathbb{Z}_p} C \xrightarrow{\mathrm{HT}_A} \omega_A \to 0.$$

Crucial here is that π_{HT} allows one to extend this pointwise consideration to the universal situation: If $\omega_{\mathcal{A}}$ denotes the conormal sheaf to the universal abelian variety $\mathcal{A} \to \mathcal{X}_{\mathfrak{c},\Gamma_0^*(p^n)}(\epsilon)_a$ and $q : \mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathfrak{c},\Gamma_0^*(p^n)}(\epsilon)_a$ is the forgetful map, then there is a canonical isomorphism

$$q^*\omega_A = \pi_{\mathrm{HT}}^*\,\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\mathcal{O}(1).$$

If F splits in L, then $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\mathcal{O}(1)$ is identified with the direct sum $\oplus_\Sigma\mathcal{O}(1)$ on $(\mathbb{P}^1)^\Sigma$, and using canonical sections of $\mathcal{O}(1)$ near $(0:1) \in \mathbb{P}^1$, we obtain a canonical section $\mathfrak{s}$ of $q^*\omega_{\mathcal{A}}$ which is a geometric incarnation of the section $\alpha(1,0)$ considered above. In general we work with a canonical section of $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\mathcal{O}(1)$, an instance of Scholze's "fake Hasse invariants" from [Sch15].

In the elliptic case, we had a canonical parameter $\mathfrak{z}$. In the Hilbert case, $\mathfrak{z}$ is now simply the restriction of π_{HT} to a function $\mathfrak{z} : \mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \to \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\hat{\mathbb{G}}_a$ where $\hat{\mathbb{G}}_a \subseteq \mathbb{P}^1$ is the closed unit ball around $(0:1) \in \mathbb{P}^1$. When F splits in L, via the canonical decomposition $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\hat{\mathbb{G}}_a = \mathbb{G}_a^\Sigma$ this can be interpreted as a collection of functions $\mathfrak{z} = (\mathfrak{z}_v)_{v\in\Sigma}$ in $\mathcal{O}^+(\mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a)$.

In order to define p-adic families of Hilbert modular forms, let $\mathcal{W}^*$ denote the weight space for G^* (cf. Defn. 6.1) and let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a bounded smooth weight. In §6, we use the sousperfectoid adic space

$$\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a := \mathcal{X}_{\mathfrak{c},\Gamma^*(p^\infty)}(\epsilon)_a \times_L \mathcal{U}$$

to define the sheaf of $\mathfrak{c}$-polarised geometric Hilbert modular forms of weight κ on $\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma_0^*(p)}(\epsilon)_a$ as

$$\omega^\kappa_{1,\mathfrak{c}} := \left\{ f \in \mathcal{O}_{\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a} \middle| \gamma^*f = \kappa^{-1}(c\mathfrak{z}+d)f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0^*(p) \right\}.$$

Here $\epsilon > 0$ is such that for any element of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0^*(p)$, we can make sense of $\kappa(c\mathfrak{z}+d)$ as an invertible function on $\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a$ (Defn. 6.4). We describe the variation of this in families, and a local version giving an overconvergent automorphic bundle $\omega^\kappa_{G^*}$on $\mathcal{X}(\epsilon)$. We also have integral versions of these spaces given by simply replacing $\mathcal{O}_{\mathcal{X}}$ with $\mathcal{O}^+_{\mathcal{X}}$ in the above definition. In Thm. 7.14 we use the canonical sections of $\omega_{\mathcal{A}}$ at infinite level to define a comparison isomorphism to the sheaf of Hilbert modular forms defined in [AIP16a].

1.3.2. *Hilbert modular varieties for G at infinite level.* For arithmetic applications, it is desirable to have a version of this theory for arithmetic Hilbert modular forms, that is for the group G. For example, these are objects that arise in modularity of elliptic curves over totally real fields.

Towards this goal, we pass from G^* to G and discuss in §8 the perfectoid Hilbert modular variety at infinite level for G and the corresponding Hodge–Tate period map. In the case of $L = \mathbb{C}_p$, this is a special case of the construction of Shen of perfectoid Shimura varieties of abelian type [She17]. For Hilbert modular varieties, this is easy to extend to general perfectoid fields following the methods of [Sch15], as we shall describe.

In contrast to the definition in [AIP16a] by descent from G^*, this allows us to define the sheaf of p-adic overconvergent arithmetic Hilbert modular forms without reference to G^*.

While this definition is ultimately quite simple, in order to explain why this is the correct definition, it is important for us to work out the geometric relation between the perfectoid Hilbert modular varieties for G and G^* rather explicitly, in particular keeping close track of all of the relevant Galois actions. At tame level, this is easy: let $\mathcal{X}$ and $\mathcal{X}_G$ be the adic analytifications of the tame $\mathfrak{c}$-polarised Hilbert modular varieties for G^* and G respectively. The natural map $\mathcal{X} \to \mathcal{X}_G$ can then be described as the quotient under the action of $\mathcal{O}_F^{\times,+}$ on the polarisation, which factors through the action of a finite group $\Delta(N)$.

For level at p, however, the condition defining level structures for G^* is *not* preserved by the action of $\mathcal{O}_F^{\times,+}$, so there is no longer a full polarisation action on the spaces $\mathcal{X}_{\Gamma^*(p^n)}$. We therefore work with an auxiliary 'mixed' moduli problem, and work with the spaces $\mathcal{X}_{\Gamma(p^n)}$ relatively parametrising G-level structures over the space $\mathcal{X}$ arising from G^*. In the limit, combined with the Weil pairing these give rise to a perfectoid space $\mathcal{X}_{\Gamma(p^\infty)}$ with a canonical $\mathbb{Z}_p^\times$-torsor

$$\mathcal{X}_{\Gamma^*(p^\infty)} \times \mathcal{O}_p^\times \longrightarrow \mathcal{X}_{\Gamma(p^\infty)}. \tag{1.2}$$

Now, on $\mathcal{X}_{\Gamma(p^\infty)}$, we *do* get an $\mathcal{O}_F^{\times,+}$ action on polarisations. At infinite level, this extends to an action of a profinite group $\Delta(p^\infty N)$, which makes the morphism $\mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{X}_{G,\Gamma(p^\infty)}$ into a pro-étale $\Delta(p^\infty N)$-torsor. We thus obtain a morphism of pro-étale torsors (cf. (8.7))

$$\begin{array}{ccc} \mathcal{X}_{\Gamma(p^\infty)} & \xrightarrow{\Delta(p^\infty N)} & \mathcal{X}_{G,\Gamma(p^\infty)} \\ {\scriptstyle \Gamma_0(p^n)}\downarrow & & \downarrow{\scriptstyle \mathrm{P}\Gamma_0(p^n)} \\ \mathcal{X}_{\Gamma_0(p^n)} & \xrightarrow{\Delta(N)} & \mathcal{X}_{G,\Gamma_0(p^n)}. \end{array} \tag{1.3}$$

The diagonal map is also a pro-étale torsor for some group $E(p^n)$, as we shall discuss in §8.

1.3.3. *Hilbert modular forms for G.* In §9, we use the above to define arithmetic Hilbert modular forms. Let $\mathcal{W}$ be the weight space for G and let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight.

Definition 1.4. The sheaf of *arithmetic Hilbert modular forms of weight κ* on $\mathcal{X}_{G,\mathfrak{c},\mathcal{U}}(\epsilon)$ is

$$\omega_G^\kappa = \left\{ f \in \mathcal{O}(\mathcal{X}_{G,\mathfrak{c},\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a) \middle| \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d) w_\kappa(\det\gamma) f \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{P}\Gamma_0(p) \right\}.$$

Its global sections $M_\kappa^G(\Gamma_0(p), \epsilon, \mathfrak{c})$ form the space of $\mathfrak{c}$-polarised ϵ-overconvergent arithmetic Hilbert modular forms. We also have an integral subsheaf $\omega_G^{\kappa,+}$ by instead using the $\mathcal{O}^+$-sheaf.

Our approach yields various natural alternative definitions; for example, we could instead use the torsor $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{G,\Gamma_0(p)}(\epsilon)_a$ to define forms for G. In §9, we show that these alternatives (see Defn. 9.1) are all equivalent to the one given above.

1.3.4. *Comparison to other definitions.* Using the canonical section $\mathfrak{s}$, we obtain a comparison isomorphism to the Hilbert modular forms of Andreatta–Iovita–Pilloni in Thms. 7.14 and 9.12.

Theorem 1.5. *There is a natural Hecke-equivariant isomorphism between $\omega_{G^*}^{\kappa,+}$ (resp. $\omega_G^{\kappa,+}$) and the sheaf of integral Hilbert modular forms for G^* (resp. G) defined in* [AIP16a, AIP16b].

We establish this for G^*, and prove that our modular forms for G are the descent of those for G^* under the action of $\Delta(N)$ (see Lem. 9.6); from this we we obtain the analogous result for G, as the forms of [AIP16a] are *defined* via this descent.

1.4. **Acknowledgements.** We would like to thank Christian Johansson for answering all our questions and for many helpful conversations. We also thank David Hansen, Vincent Pilloni and Peter Scholze for useful discussions. We would like to thank Lucas Mann and Mingjia Zhang for their comments on an earlier draft. Finally, we thank the referee for very useful comments.

The work was supported by the following grants: EPSRC Doctoral Prize Fellowship at University College London, [EP/N509577/1] (Birkbeck); The EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London, EPSRC [EP/L015234/1] (Heuer); and a Heilbronn Research Fellowship (Williams).

1.5. **Notation.** We fix a rational prime p and a perfectoid field extension L of $\mathbb{Q}_p^{\mathrm{cyc}}$. For instance, we could take $L = \mathbb{Q}_p^{\mathrm{cyc}}$, or any complete algebraically closed extension of $\mathbb{Q}_p$.

We use adic spaces in the sense of Huber [Hub13], and in particular the notion of smooth adic spaces from [Hub13, Definition 1.6.5]. By a rigid space over L, we mean an adic space of topologically finite type over $\mathrm{Spa}(L, \mathcal{O}_L)$. We use the pro-étale site of a smooth adic space in the sense of [Sch13].

The letter X typically refers to modular curves and Hilbert modular varieties. We typically use latin letters X to refer to schemes, gothic letters $\mathfrak{X}$ to refer to formal schemes, and calligraphic

letters $\mathcal{X}$ to refer to analytic adic spaces (typically over L), like rigid and perfectoid spaces. If X is a modular variety of some tame level, we specify the level at p of a modular variety by a subscript on the appropriate X. We will use a superscript * to denote the minimal compactification X^*.

Lastly, if Γ is any profinite set we also write Γ for the associated profinite perfectoid space $\mathrm{Spa}(\mathrm{Maps}_{\mathrm{cts}}(\Gamma, L), \mathrm{Maps}_{\mathrm{cts}}(\Gamma, \mathcal{O}_L))$ when this is clear from the context. If Γ is a profinite group, this will be a group object in perfectoid spaces over L.

2. Perfectoid modular curves and the Hodge–Tate period map

In this section we review the modular curve at infinite level and the Hodge–Tate period map, and discuss the open subspaces which we are going to use to define p-adic elliptic modular forms.

2.1. **Modular curves and their canonical and anticanonical loci.** Let N be an integer coprime to p. Let X be the modular curve over L of some tame level Γ^p at N such that the corresponding moduli problem is representable by a scheme, e.g. $\Gamma(N)$ or $\Gamma_1(N)$ for $N \geq 3$. Similarly, we let $\Gamma_0(p) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ denote the usual upper triangular Iwahori subgroup, corresponding to the choice of an order p sub-group-scheme of our elliptic curves.

We denote by $\mathcal{X}$ the rigid analytification, considered as an adic space.[a] The space $\mathcal{X}$ represents the moduli functor that sends any adic space $S \to \mathrm{Spa}(L, \mathcal{O}_L)$ to isomorphism classes of pairs (E, μ) where E is an elliptic curve over $\mathcal{O}_S(S)$ with Γ^p-structure μ (see [Heua, Lemma 2.3]).

Let X^* be the compactification of X, with analytification $\mathcal{X}^*$. The divisor of cusps $\mathcal{X}^* \backslash \mathcal{X}$ becomes a finite set of closed points after adding a primitive N-th root to L.

For any $0 \leq \epsilon < 1$ with $|p|^\epsilon \in |L|$, we denote by $\mathcal{X}^*(\epsilon) \subseteq \mathcal{X}^*$ the open subspace of the modular curve where $|\widetilde{\mathrm{Ha}}| \geq |p|^\epsilon$, where $\widetilde{\mathrm{Ha}}$ is any local lift of the Hasse invariant. For any analytic adic space $\mathcal{Y} \to \mathcal{X}^*$, we write $\mathcal{Y}(\epsilon) \subseteq \mathcal{Y}$ for the preimage of $\mathcal{X}^*(\epsilon) \subseteq \mathcal{X}^*$. We call the elliptic curves parametrised by this open subspace *ϵ-nearly ordinary*.

Let $\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim \mathcal{X}^*_{\Gamma(p^n)}$ be the perfectoid modular "curve" at infinite level as defined in [Sch15]. We in particular have the open subspace $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon) \sim \varprojlim \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)$.

Recall that for any $n \in \mathbb{Z}_{\geq 1}$, the modular curve $X_{\Gamma_0(p^n)} \to X$ of level $\Gamma_0(p^n)$ relatively represents the choice of a cyclic rank p^n subgroup scheme $D_n \subseteq E[p^n]$. If $0 \leq \epsilon < 1/(p+1)p^{n-2}$ then by Lubin's theory of the canonical subgroup, any elliptic curve corresponding to a morphism $S \to \mathcal{X}(\epsilon)$ admits a canonical cyclic subgroup scheme $H_n \subseteq E[p^n]$ of rank p^n, which in the case of good reduction reduces to the kernel of the n-th iterate of Frobenius on E modulo $p^{1-\epsilon}$. This defines a canonical section $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon) \leftarrow \mathcal{X}(\epsilon)$ which in fact extends to the cusps. As a consequence, for $n = 1$, the space $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)$ decomposes into two open and closed components

$$\mathcal{X}^*_{\Gamma_0(p)}(\epsilon) \quad = \quad \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_c \quad \dot\sqcup \quad \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a,$$

the first of which (away from the cusps) parametrises triples (E, α, H_1) with α a tame level and H_1 the canonical subgroup, while the second parametrises (E, α, D_n) with $D_n \subseteq E[p]$ a cyclic rank p subgroup such that $D_n \cap H_1 = 0$. The two components are called the canonical and the anticanonical locus, respectively. At infinite level, these two components, via pullback, give rise to canonical and anticanonical loci of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)$ respectively:

$$\begin{array}{ccccc} \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon) & = & \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_c & \dot\sqcup & \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_0(p)}(\epsilon) & = & \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_c & \dot\sqcup & \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a. \end{array}$$

For any perfectoid $(\mathbb{Q}_p^{\mathrm{cyc}}, \mathbb{Z}_p^{\mathrm{cyc}})$-algebra (R, R^+), the (R, R^+)-points of $\mathcal{X}_{\Gamma(p^\infty)}$ are in functorial 1-1 correspondence with isomorphism classes of triples (E, μ, α), where E is an elliptic curve over R, μ is a Γ^p-structure, and $\alpha : \mathbb{Z}_p^2 \xrightarrow{\sim} T_pE$ is a trivialisation of the Tate module (see [Heua, Cor. 3.2]). We have an action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\mathcal{X}_{\Gamma(p^\infty)}$ given by

$$\gamma \cdot (E, \mu, \alpha) = (E, \mu, \alpha \circ \gamma^\vee), \quad \gamma^\vee = \det(\gamma)\gamma^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \text{ for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p). \tag{2.1}$$

[a] We note that this is the only way in which our notation deviates from that in [Sch15, §III], where $\mathcal{X}$ denotes the good reduction locus.

For notational convenience we also fix the following.

Definition 2.1. Let E be an ordinary elliptic curve. Then E has canonical cyclic subgroups H_n of rank p^n for all n. The *canonical p-divisible subgroup of E* is $H = (H_n)_{n\in\mathbb{N}} \subseteq E[p^\infty]$.

2.2. **The Hodge–Tate period map around $0 \in \mathbb{P}^1$.** We recall how the canonical and anti-canonical loci behave under the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1.$$

By [Sch15, Lem. III.3.6], the preimage of $\mathbb{P}^1(\mathbb{Z}_p)$ under π_{HT} is given by the closure of the ordinary locus $\mathcal{X}^*_{\Gamma(p^\infty)}(0)$. After removing the cusps, this parametrises isomorphism classes of triples (E, μ, α) as above where E has potentially semistable or good ordinary reduction in every fibre. Write e_1, e_2 for the standard basis of $\mathbb{Z}_p^2$. Away from cusps, the preimage $\pi_{\mathrm{HT}}^{-1}(\infty)$ of $\infty = (1:0)$ parametrises triples where moreover $\alpha(e_1)$ generates the canonical p-divisible subgroup.

Instead of using the canonical locus, we shall work with the anticanonical locus $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, which by contrast is sent by π_{HT} to neighbourhoods of points of the form $(b:1)$ for $b \in \mathbb{Z}_p$.

In order to define overconvergent modular forms on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ of weight κ for $\epsilon > 0$, we need to make sense of the expression $\kappa(c\pi_{\mathrm{HT}}(x) + d)$ for $x \in \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. To account for the overconvergence, we therefore need to consider open neighboorhods of these points:

Definition 2.2. Let $B_0(\mathbb{Z}_p : 1) \subseteq \mathbb{P}^1(\mathbb{Z}_p) \subseteq \mathbb{P}^1$ be the subspace of points of the form $(a:1)$ for $a \in \mathbb{Z}_p$ considered as a profinite adic space. For any $0 < r \leq 1$ and any compact open subspace $U \subseteq \mathbb{Z}_p$, let $B_r(U:1) \subseteq \mathbb{P}^1$ be the subspace defined as the union of all closed balls of radius r around points $(a:1) \in \mathbb{P}^1(\mathbb{Z}_p)$ with $a \in U$

Definition 2.3. Let z be the parameter on $\mathbb{P}^1$ at 0 arising from the canonical isomorphism of schemes $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{P}^1\backslash\{\infty\}$, $z \mapsto (z:1)$ and let $\mathfrak{z} := \pi_{\mathrm{HT}}^* z$. It is easy to see that $B_r(\mathbb{Z}_p : 1) \subseteq \mathbb{A}^1$ is closed both under the additive group structure as well as the multiplicative monoid structure.

Lemma 2.4. *The action of $\Gamma_0(p)$ fixes $B_r(\mathbb{Z}_p : 1) \subseteq \mathbb{P}^1$. In terms of the parameter z, we have*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}. \tag{2.2}$$

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$. Then inside $\mathbb{P}^1$ we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z:1) = (az+b : cz+d) = (\gamma z : 1)$. Moreover, $|cz+d| = 1$ on $B_r(\mathbb{Z}_p : 1)$ since $|z| \leq 1$ (as $r \leq 1$) , $c \in p\mathbb{Z}_p$, $d \in \mathbb{Z}_p^\times$. Consequently, since $(cz+d)^{-1} = d^{-1}\sum_{n\geq 0}(cd^{-1}z)^n \in B_r(\mathbb{Z}_p : 1)$, the fact that $B_r(\mathbb{Z}_p : 1)$ is closed under multiplication and addition implies that also $\gamma z = (az+b)(cz+d)^{-1} \in B_r(\mathbb{Z}_p : 1)$ as desired. □

Remark 2.5. In the definition of modular forms in [CHJ17], the automorphic factor features the term $(b\mathfrak{z} + d)$, since in their notation – where z is a parameter for a neighbourhood of $\infty \in \mathbb{P}^1$ – the action of $\Gamma_0(p)$ is given by $z \mapsto (b+dz)/(a+cz)$. In switching from the canonical to the anticanonical locus, we instead get $(c\mathfrak{z} + d)$ as in the complex case (see also Rem. 3.23).

The following proposition implies that for any weight κ, there is an $\epsilon > 0$ such that for any $x \in \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ and any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, the factor of automorphy $\kappa(c\mathfrak{z}(x) + d)$ converges.

Proposition 2.6. *Let $0 \leq r < 1$. Then for $0 \leq \epsilon \leq r/2$ if $p \geq 5$ or $\epsilon \leq r/3$ if $p = 3$, or $\epsilon \leq r/4$ if $p = 2$, we have $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) \subseteq B_r(\mathbb{Z}_p : 1)$.*

Proof. Away from the cusps, this is a special case of Prop. 5.18 below. For the cusps, the statement is clear since these are contained in the ordinary locus and are thus sent to $\mathbb{P}^1(\mathbb{Q}_p)$. □

3. Overconvergent elliptic modular forms

In this section we define line bundles of p-adic modular forms of weight κ, where κ is a smooth bounded weight. Following [CHJ17] with our slightly modified setup, these bundles are defined using the structure sheaf of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ by taking invariants under a group action with a factor of automorphy to descend to finite level, mirroring the definition of complex modular forms.

We first explain what we mean by a smooth bounded weight:

Definition 3.1. The *weight space* for GL_2 is $\mathcal{W} := \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\mathrm{ad}}_\eta \times_{\mathbb{Q}_p} L$. A *smooth weight* over L is a smooth adic space $\mathcal{U}$ over a perfectoid extension of L together with a map $\mathcal{U} \to \mathcal{W}$. A smooth weight is *bounded* if its image in $\mathcal{W}$ is contained in some affinoid subspace of $\mathcal{W}$.

3.1. **Sousperfectoid spaces.** We would like to define sheaves of families of modular forms of weight $\mathcal{U}$ to be functions on $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \times \mathcal{U}$. In order to obtain a sheaf, we need to make sure that the latter fibre product exists as an adic space. For this we use the language of sousperfectoid spaces, which we briefly recall from [HK, Seection 7] and [SW, Section 6.3]. Their technical importance stems from Prop. 3.3.

Definition 3.2. (1) A complete Tate $\mathbb{Z}_p$-algebra R is called *sousperfectoid* if there is a perfectoid R-algebra $\widetilde{R}$ such that $R \hookrightarrow \widetilde{R}$ splits in the category of topological R-modules.
(2) A Huber pair (R, R^+) is called *sousperfectoid* if R is sousperfectoid.
(3) An adic space is called *sousperfectoid* if it can be covered by affinoid open subspaces of the form $\mathrm{Spa}(R, R^+)$ where R is sousperfectoid.

Proposition 3.3. [SW, Prop. 6.3.4] *Any sousperfectoid Huber pair (R, R^+) is stably uniform. In particular, $\mathrm{Spa}(R, R^+)$ is a sheafy adic space.*

Corollary 3.4. *Let $\mathcal{X}$ be a perfectoid space over L and let $\mathcal{Y}$ be a rigid space smooth over a perfectoid extension of L'/L. Then the fibre product $\mathcal{X} \times_L \mathcal{Y}$ exists as a sousperfectoid adic space.*

Proof. By [Hub13, Cor. 1.6.10], the smooth rigid space $\mathcal{Y}$ can be covered by open subspaces which are étale over some disc $B = \mathrm{Spa}(L'\langle X_1, \dots, X_n\rangle)$. Since the fibre product of perfectoid spaces is perfectoid, we may without loss of generality assume that $L = L'$, and that $\mathcal{X} = \mathrm{Spa}(S, S^+)$ is affinoid perfectoid. The fibre product $\mathcal{X} \times_L B$ then exists and is sousperfectoid because the algebra $S\langle X_1, \dots, X_n\rangle$ is sousperfectoid by [SW, Prop. 6.3.3.(i) and (iii)]. The fibre product $\mathcal{X} \times_L \mathcal{Y} = (\mathcal{X} \times_L B) \times_B \mathcal{Y}$ now exists and is sousperfectoid because algebras étale over a sousperfectoid algebra are again sousperfectoid (Prop. 6.3.3.(ii) *op. cit.*). □

Corollary 3.5. *If $\mathcal{U}$ is a smooth adic space over L, then $\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a := \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \times_L \mathcal{U}$ exists as a sousperfectoid adic space. Moreover, if we define $\mathcal{X}^*_{\mathcal{U},\Gamma(p^n)}(\epsilon)_a := \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \times_L \mathcal{U}$, then*

$$\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \sim \varprojlim_{n\in\mathbb{N}} \mathcal{X}^*_{\mathcal{U},\Gamma(p^n)}(\epsilon)_a.$$

Proof. The first part is immediate from the last corollary. The second part follows from the observation that when $(A_n)_{n\in\mathbb{N}}$ is a direct system of Tate algebras (by which we mean a Huber pair with a topologically nilpotent unit), and A_∞ is a Tate algebra with compatible morphisms $A_n \to A_\infty$ such that $\varinjlim A_n \subseteq A_\infty$ has dense image, and B is a Tate algebra over A_1, then $\varinjlim (A_n \hat{\otimes}_{A_1} B) \subseteq A_\infty \hat{\otimes}_{A_1} B$ has dense image by pointwise approximation. □

Lemma 3.6 ([KL, Thm. 8.2.3]). *Let $\mathcal{Y}$ be a seminormal adic space (see [KL, Defn. 3.7.1]), for example a smooth rigid space. Let $v : \mathcal{Y}_{\mathrm{pro\acute{e}t}} \to \mathcal{Y}_{\mathrm{an}}$ be the natural map. Then $v_*\widehat{\mathcal{O}}^+_{\mathcal{Y}_{\mathrm{pro\acute{e}t}}} = \mathcal{O}^+_{\mathcal{Y}}$.*

Proof. By [KL, Thm. 8.2.3], we have $v_*\widehat{\mathcal{O}}_{\mathcal{Y}_{\mathrm{pro\acute{e}t}}} = \mathcal{O}_{\mathcal{Y}_{\mathrm{an}}}$. Using the adjunction morphism of v, we thus have inclusions $\mathcal{O}^+_{\mathcal{Y}} \subseteq v_*\widehat{\mathcal{O}}^+_{\mathcal{Y}_{\mathrm{pro\acute{e}t}}} \subseteq \mathcal{O}_{\mathcal{Y}}$. On the other hand, for any affinoid $V \subseteq \mathcal{Y}$, we clearly have $v_*\widehat{\mathcal{O}}^+_{\mathcal{Y}_{\mathrm{pro\acute{e}t}}}(V) \subseteq v_*\widehat{\mathcal{O}}_{\mathcal{Y}_{\mathrm{pro\acute{e}t}}}(V)^\circ = \mathcal{O}_{\mathcal{Y}}(V)^\circ$. Since $\mathcal{Y}$ is a rigid space, we have $\mathcal{O}_{\mathcal{Y}}(V)^\circ = \mathcal{O}^+_{\mathcal{Y}}(V)$, which shows $v_*\widehat{\mathcal{O}}^+_{\mathcal{Y}_{\mathrm{pro\acute{e}t}}} \subseteq \mathcal{O}^+_{\mathcal{Y}}$. □

Lemma 3.7. *Let $\mathcal{Y}$ be an affinoid adic space over L that is either a smooth rigid space or a perfectoid space. Let Γ be a profinite group. Let $\mathcal{X} \in \mathcal{Y}_{\mathrm{pro\acute{e}t}}$ be an affinoid perfectoid pro-étale Γ-torsor. Let $\mathcal{U}$ be a smooth adic space over L, set $\mathcal{X}_\mathcal{U} := \mathcal{X} \times_L \mathcal{U}$ and $\mathcal{Y}_\mathcal{U} := \mathcal{Y} \times_L \mathcal{U}$, and denote the induced map by $h : \mathcal{X}_\mathcal{U} \to \mathcal{Y}_\mathcal{U}$. Then*

$$(h_*\mathcal{O}^+_{\mathcal{X}_\mathcal{U}})^\Gamma = \mathcal{O}^+_{\mathcal{Y}_\mathcal{U}} \quad \text{and} \quad (h_*\mathcal{O}_{\mathcal{X}_\mathcal{U}})^\Gamma = \mathcal{O}_{\mathcal{Y}_\mathcal{U}}.$$

Proof. As the statement is local on $\mathcal{Y}_\mathcal{U}$, it suffices to check that for an affinoid open $V \subseteq \mathcal{Y}$ with affinoid perfectoid preimage $W = h^{-1}(V)$ we have $\mathcal{O}^+_{\mathcal{X}_\mathcal{U}}(V \times \mathcal{U})^\Gamma = \mathcal{O}^+_{\mathcal{Y}_\mathcal{U}}(W \times \mathcal{U})$. Since

by our assumptions $\mathcal{Y}$ is stably uniform, [CHJ17, Lem. 2.23 (2)] reduces this to checking that $(h_*\mathcal{O}^+_{\mathcal{X}})^\Gamma = \mathcal{O}^+_{\mathcal{Y}}$. To see this, we first treat the case that $\mathcal{Y}$ is a smooth rigid space. Then in the pro-étale site $\mathcal{Y}_{\text{proét}}$ in the sense of [Sch13], we have the structure sheaf $\mathcal{O}^+_{\mathcal{Y}_{\text{proét}}}$ as well as the completed structure sheaf $\widehat{\mathcal{O}}^+_{\mathcal{Y}_{\text{proét}}}$. For the affinoid perfectoid space $\mathcal{X}$, we have $\mathcal{O}^+_{\mathcal{X}}(\mathcal{X}) = \widehat{\mathcal{O}}^+_{\mathcal{Y}_{\text{proét}}}(\mathcal{X})$. The Cartesian diagram expressing $\mathcal{X} \to \mathcal{Y}$ as a pro-étale Γ-torsor then shows that we have

$$\mathcal{O}^+_{\mathcal{X}}(\mathcal{X})^\Gamma = \widehat{\mathcal{O}}^+_{\mathcal{Y}_{\text{proét}}}(\mathcal{X})^\Gamma = \widehat{\mathcal{O}}^+_{\mathcal{Y}_{\text{proét}}}(\mathcal{Y}).$$

The first part of the Lemma now follows from Lem. 3.6. The second follows by inverting p.

If $\mathcal{Y}$ is a perfectoid space, the same argument works in the pro-étale site $\mathcal{Y}_{\text{proét}}$ of [Sch]. □

Proposition 3.8. *Let $\mathcal{U}$ be a smooth adic space over a perfectoid field extension L' of L. For any $n \geq 1$ denote by $h : \mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$ the natural map. Then*

$$(h_*\mathcal{O}^+_{\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a})^{\Gamma_0(p^n)} = \mathcal{O}^+_{\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a} \quad \text{and} \quad (h_*\mathcal{O}_{\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a})^{\Gamma_0(p^n)} = \mathcal{O}_{\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a}.$$

For the proof, we explain how to deal with the boundary, which was not treated in [CHJ17].

Proof. After base-change to L', we may without loss of generality assume that $L = L'$.

Over the open subspace away from the cusps, the map $h : \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$ is a pro-étale $\Gamma_0(p^n)$-torsor for the action defined in (2.1). By Lem. 3.7, we thus have

$$(h_*\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a})^{\Gamma_0(p^n)} = \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a}.$$

We are left to extend this to the cusps. Let us first look at the case that $\mathcal{U}$ is a single point. For this we can use Tate curve parameter discs as discussed in [Heua]: For any geometric point c in the boundary of $\mathcal{X}^*$, there is an integer $d|N$ (depending on the tame level structure and the presence of unit roots in L) such that there is an open immersion $D \times \mu_d \hookrightarrow \mathcal{X}^*$ where $D \subseteq L\langle q\rangle$ is the open disc defined by $|q| < 1$, such that the image of the origin contains c. For $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ there is then also a Tate curve parameter disc $D \times \mu_d \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$. The induced map over $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}^*$ is $D \to D, q \mapsto q^{p^n}$ by Prop. 2.10 *op. cit.* Equivalently, we may rewrite this as the open disc $D_n \subseteq \mathrm{Spa}(L\langle q^{1/p^n}\rangle)$. By taking tilde-limits, we obtain a perfectoid disc $D_\infty \sim \varprojlim D_n \subseteq \mathrm{Spa}(L\langle q^{1/p^\infty}\rangle)$. By [Heua, Thm. 3.8], there is then a Cartesian diagram

$$\begin{array}{ccc} \Gamma_0(p^\infty) \times D_\infty \times \mu_d & \longrightarrow & D_n \times \mu_d \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon) \end{array}$$

where $\Gamma_0(p^\infty)$ is the profinite perfectoid group of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$. By Thm. 3.21 *op. cit.*, the $\Gamma_0(p^n)$-invariance of a function f on $\Gamma_0(p^\infty) \times D_\infty \times \mu_d$ now means precisely the following: first, the $\Gamma_0(p^\infty)$-invariance means that f comes from a function on $D_\infty \times \mu_d$ via pullback along the projection $\Gamma_0(p^\infty) \times D_\infty \times \mu_d \to D_\infty \times \mu_d$. It is thus of the form $f \in \mathcal{O}^+(D_\infty \times \mu_d) = \mathcal{O}_L[\zeta_d][[q^{1/p^\infty}]]$. Second, the remaining $\Gamma_0(p^n)/\Gamma_0(p^\infty) = p^n\mathbb{Z}_p$-action is the one which sends $q^{1/p^m} \to \zeta_{p^m}^h q^{1/p^m}$ for all $m \in \mathbb{N}$ and $h \in p^n\mathbb{Z}_p$. For f to be invariant under this action means that $f \in \mathcal{O}^+(D_n \times \mu_d) = \mathcal{O}_L[\zeta_d][[q^{1/p^n}]]$, as desired.

For general weight $\mathcal{U}$, we may without loss of generality assume that $\mathcal{U}$ is affinoid. The same argument then still works, adding a fibre product with $\mathcal{U}$ everywhere in the above and working with q-expansions in $\mathcal{O}^+(\mathcal{U})[\zeta_d][[q^{1/p^n}]]$ instead. □

Remark 3.9. As pointed out to us by the referee, one can also use the rigid analytic Riemann Hebbarkeitssatz for normal rigid spaces due to Bartenwerfer [Bar76] and Lütkebohmert [Lüt74, Satz 1.6] to extend over the boundary. One advantage of the perspective taken above is that it explicitly describes the q-expansions associated to p-adic modular forms.

3.2. **Overconvergent modular forms.**

Definition 3.10. For any rational $0 < r \leq 1$, let $B_r(\mathbb{Z}_p^\times : 1) \subset B_r(\mathbb{Z}_p : 1) \subset \mathbb{P}^1$ be the union of all balls $B_r(a)$ of radius r around points $(a : 1)$ with $a \in \mathbb{Z}_p^\times$. For $r = 0$, we instead let

$B_0(\mathbb{Z}_p^\times : 1) := \mathbb{Z}_p^\times$, where we recall that we mean by this the perfectoid space associated to the profinite group $\mathbb{Z}_p^\times$.

The adic space $B_r(\mathbb{Z}_p^\times : 1)$ inherits the structure of an adic group from $\mathbb{G}_m = \mathbb{P}^1\backslash\{0,\infty\} \subseteq \mathbb{P}^1$. Note that for $r < 1$, on $\mathbb{Z}_p$-points we have $B_r(\mathbb{Z}_p^\times : 1)(\mathbb{Z}_p) = \mathbb{Z}_p^\times$ compatible with the group structure. We therefore regard $B_r(\mathbb{Z}_p^\times : 1)$ as an adic analytic thickening of $\mathbb{Z}_p^\times$.

Definition 3.11. (1) In the following, we will also denote by $\mathbb{G}_a$ the adic analytification of the corresponding scheme over L. Its underlying adic space is $\mathbb{A}^{1,\mathrm{an}}$, and it represents the functor sending an adic space $\mathcal{Z}$ over $(L, \mathcal{O}_L)$ to $\mathcal{O}(\mathcal{Z})$.

(2) We analogously define the adic group $\mathbb{G}_m$ as the adic analytification of the corresponding scheme over L. It represents the functor sending an adic space $\mathcal{Z}$ over $(L, \mathcal{O}_L)$ to $\mathcal{O}(\mathcal{Z})^\times$.

(3) Denote by $\hat{\mathbb{G}}_a$ the adic generic fibre of the formal completion of the $\mathcal{O}_L$-scheme $\mathbb{G}_{a,\mathcal{O}_L}$; then $\hat{\mathbb{G}}_a \subseteq \mathbb{G}_a$ is an open subgroup, given by the closed ball of radius 1 around the origin $0 \in \mathbb{G}_a$. It represents the functor sending an adic space $\mathcal{Z}$ over $(L, \mathcal{O}_L)$ to $\mathcal{O}^+(\mathcal{Z})$.

(4) Similarly, let $\hat{\mathbb{G}}_m$ be the adic generic fibre of the formal completion of $\mathbb{G}_{a,\mathcal{O}_L}$; then $\hat{\mathbb{G}}_m \subseteq \mathbb{G}_m$ is an open subgroup, given by the closed ball of radius 1 around the origin $1 \in \mathbb{G}_m$. It represents the functor sending an adic space $\mathcal{Z}$ over $(L, \mathcal{O}_L)$ to $\mathcal{O}^+(\mathcal{Z})^\times$.

Any continuous character $\kappa : \mathbb{Z}_p^\times \to L^\times$ has a geometric incarnation as a morphism of adic spaces $\kappa : \mathbb{Z}_p^\times \to \hat{\mathbb{G}}_m$. Indeed, by the universal property of $\mathbb{G}_m$, any such morphism corresponds to an element of $\mathcal{O}(\mathbb{Z}_p^\times)^\times = \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L)^\times = \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L^\times)$. Any such κ has an analytic continuation to $B_r(\mathbb{Z}_p^\times : 1)$ for small enough r. In fact, this holds more generally: let $\mathcal{U}$ be any bounded smooth weight (see Defn. 3.1). This corresponds to a morphism $\kappa : \mathbb{Z}_p^\times \times \mathcal{U} \to \hat{\mathbb{G}}_m$ or, equivalently, a continuous morphism $\mathbb{Z}_p^\times \to \mathcal{O}_{\mathcal{U}}(\mathcal{U})^\times$, called the *character of* $\mathcal{U}$.

Proposition 3.12. *If $\kappa : \mathcal{U} \to \mathcal{W}$ is a bounded smooth weight, then there exists r_κ such that for $r_\kappa \geq r > 0$ there is a unique morphism*

$$\kappa^{\mathrm{an}} : B_r(\mathbb{Z}_p^\times : 1) \times \mathcal{U} \to \hat{\mathbb{G}}_m$$

such that the restriction of κ^{an} to $\mathbb{Z}_p^\times \times \mathcal{U}$ via $\mathbb{Z}_p^\times \hookrightarrow B_r(\mathbb{Z}_p^\times : 1)$, $a \mapsto (a : 1)$, is equal to κ.

Proof. If $\mathcal{U}$ is affinoid this is a special case of [Buz07, Prop. 8.3]. In general, the assumption that $\mathcal{U}$ is bounded ensures that $\mathcal{U}$ is contained in some affinoid open subspace of $\mathcal{W}$. For a precise value of r_κ see Prop. 6.3. □

Definition 3.13. For any smooth bounded weight $\kappa : \mathcal{U} \to \mathcal{W}$, let r_κ be the supremum of all r such that the proposition holds. Similarly, let $\epsilon_\kappa^{\mathrm{def}}$ be the maximum ϵ satisfying the conditions of Prop. 2.6 with respect to r_κ, then $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) \subseteq B_r(\mathbb{Z}_p : 1)$.

Recall $\mathfrak{z} := \pi_{\mathrm{HT}}^* z$ is the function on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ defined by pullback of the function z on $B_r(\mathbb{Z}_p : 1)$ from Defn. 2.3. Since π_{HT} is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant, Lem. 2.4 implies that for any $\gamma \in \Gamma_0(p)$, we have

$$\gamma^* \mathfrak{z} = \frac{a\mathfrak{z} + b}{c\mathfrak{z} + d}. \tag{3.1}$$

Using this and Prop. 2.6, we can then make the following definition:

Definition 3.14. Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight and $\epsilon_\kappa^{\mathrm{def}} > \epsilon \geq 0$. For any $c \in p\mathbb{Z}_p$, $d \in \mathbb{Z}_p^\times$, we then let $\kappa(c\mathfrak{z} + d)$ be the invertible function on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$ defined by

$$\kappa(c\mathfrak{z} + d) : \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \xrightarrow{\pi_{\mathrm{HT}} \times \mathrm{id}} B_r(\mathbb{Z}_p : 1) \times \mathcal{U} \xrightarrow{(z \mapsto cz+d) \times \mathrm{id}} B_r(\mathbb{Z}_p^\times : 1) \times \mathcal{U} \xrightarrow{\kappa^{\mathrm{an}}} \hat{\mathbb{G}}_m.$$

We can now give the definition of sheaves of overconvergent modular forms of weight κ.

Definition 3.15. For $\kappa : \mathcal{U} \to \mathcal{W}$ a bounded smooth weight, $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $0 \leq \epsilon \leq \epsilon_\kappa^{\mathrm{def}}$, we define a sheaf ω_n^κ on $\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$ by

$$\omega_n^\kappa := \left\{ f \in q_* \mathcal{O}_{\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a} \middle| \gamma^* f = \kappa^{-1}(c\mathfrak{z} + d) f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^n) \right\},$$

where we recall that $q : \mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$ denotes the projection. We also have an integral subsheaf $\omega_n^{\kappa,+}$ on $\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$ defined by using instead the $\mathcal{O}^+$-sheaf:

$$\omega_n^{\kappa,+} := \left\{ f \in q_*\mathcal{O}^+_{\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a} \middle| \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^n) \right\}.$$

For $n = 0$ and $\epsilon_\kappa/p > \epsilon \geq 0$, we use the Atkin–Lehner isomorphism $\mathrm{AL} : \mathcal{X}^*_{\Gamma_0(p)}(p\epsilon)_a \xrightarrow{\sim} \mathcal{X}^*(\epsilon)$ to define the sheaf of overconvergent p-adic modular forms ω^κ to be the sheaf $\mathrm{AL}_*\omega_1^\kappa$ on $\mathcal{X}^*_{\mathcal{U}}(\epsilon)$. We also have the $\mathcal{O}^+$-submodule of integral p-adic modular forms $\omega^{\kappa,+}$ given by $\mathrm{AL}_*\omega_1^{\kappa,+}$.

We will later see that these are all invertible sheaves. Note that for $n = \infty$, this defines sheaves of modular forms on $\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^\infty)}(\epsilon)_a$ which, by analogy to [AIP16a], we call *perfect modular forms*.

Definition 3.16. Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight and let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. We define the space of *overconvergent modular forms of weight* $\mathcal{U}$*, wild level* $\Gamma_0(p^n)$*, tame level* Γ^p *and radius of overconvergence* $0 \leq \epsilon < \epsilon_\kappa^{\mathrm{def}}$ to be the L-vector space

$$\begin{aligned} M_\kappa(\Gamma_0(p^n), \epsilon) &:= \mathrm{H}^0(\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a, \omega_n^\kappa) \\ &= \left\{ f \in \mathcal{O}(\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^\infty)}(\epsilon)_a) \middle| \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma \in \Gamma_0(p^n) \right\}, \end{aligned}$$

and the analogous space of *integral* forms to be

$$M_\kappa^+(\Gamma_0(p^n), \epsilon) := \mathrm{H}^0(\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a, \omega_n^{\kappa,+}).$$

Finally, we set $M_\kappa(\epsilon) := M_\kappa(\Gamma_0(p), p\epsilon)$, and analogously for the integral subspaces. We note that for any $n < \infty$, there is then a natural Atkin–Lehner isomorphism

$$M_\kappa^+(\Gamma_0(p^n), \epsilon) \cong M_\kappa^+(p^{-n}\epsilon).$$

One can similarly define cusp forms by working instead with $\omega_n^\kappa(-\partial)$ where ∂ denotes the boundary divisor in $\mathcal{X}^*_{\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$. As usual, one now defines an action of Hecke operators T_ℓ for $\ell \nmid Np$ and U_p via correspondences. We shall discuss this in detail in the Hilbert case in §10.

3.3. **Comparison to overconvergent modular forms of classical weights.** Recall that on $\mathcal{X}^*$ we have the conormal sheaf $\omega_{\mathcal{E}} := \pi_*\Omega^1_{\mathcal{E}|\mathcal{X}}$ of the universal semi-abelian scheme $\pi : \mathcal{E} \to \mathcal{X}^*$. For a p-level structure Γ_p of the form $\Gamma_0(p^n)$ or $\Gamma(p^n)$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we write ω_{Γ_p} for the pullback of $\omega_{\mathcal{E}}$ to $\mathcal{X}^*_{\Gamma_p}(\epsilon)_a$. In this section, we show that for characters κ of the form $x \mapsto x^k$, for $k \in \mathbb{Z}_{\geq 1}$, the sheaf ω_n^κ can be identified with $\omega^{\otimes k}_{\Gamma_0(p^n)}$. This shows that for classical weights, our definition agrees with the usual spaces of overconvergent modular forms, and contains the spaces of classical modular forms. The key to the comparison is the isomorphism of line bundles

$$\pi_{\mathrm{HT}}^*\mathcal{O}(1) = \omega_{\Gamma(p^\infty)} \tag{3.2}$$

from [Sch15, Thm. III.3.18]. We recall that on (C, C^+)-points, this has the following moduli interpretation: The C-points of the total space $\mathcal{T}(1) \to \mathbb{P}^1$ of the line bundle $\mathcal{O}(1)$ parametrise pairs (L, y) of a line $L \subseteq C^2$ together with a point $y \in C^2/L$ on the quotient. Equivalently, this is the data (φ, y) of a linear projection $\varphi : C^2 \twoheadrightarrow Q$ to a 1-dimensional C-vector space Q and a point $y \in Q$. We sometimes just write this as the point y if Q is clear from context. Using this description of $\mathcal{O}(1)$, one can now illustrate equation (3.2) as follows.

Lemma 3.17. *Let x be a (C, C^+)-point of $\mathcal{X}_{\Gamma(p^\infty)}$, corresponding to data $(E, \mu, \alpha : \mathbb{Z}_p^2 \xrightarrow{\sim} T_pE)$. Let $\mathrm{HT}_E : T_pE \to \omega_E$ be the Hodge–Tate map of E. In terms of the total spaces, the isomorphism $\omega_{\Gamma(p^\infty)} = \pi_{\mathrm{HT}}^*\mathcal{O}(1)$ is given in the fibre of x by the morphism*

$$(E,\ \alpha,\ \eta \in \omega_E) \mapsto (\mathrm{HT} : T_pE \otimes_{\mathbb{Z}_p} C \to \omega_E,\ \eta \in \omega_E).$$

Following [CHJ17], we now compare our sheaf of modular forms ω^κ for $\kappa = \mathrm{id}$ to the bundle of differentials $\omega_{\mathcal{E}}$ by using an explicit trivialisation of $\mathcal{O}(1)$ over $B_r(\mathbb{Z}_p : 1)$:

Definition 3.18. Let s be the global section $\mathbb{P}^1 \to \mathcal{T}(1)$ given by

$$(x : y) \mapsto \left(C^2 \to C^2/\left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right\rangle,\ \ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right\rangle\right),$$

where the second component lies in $C^2/\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \rangle$. This section is non-vanishing away from the point $(1 : 0) = \infty \in \mathbb{P}^1$, and in particular it is invertible over $B_r(\mathbb{Z}_p : 1) \subseteq \mathbb{P}^1$.

We need to compute the action of $\Gamma_0(p)$ on s over $B_r(\mathbb{Z}_p : 1) \subseteq \mathbb{G}_a \subseteq \mathbb{P}^1$ in terms of the parameter $(z : 1)$. For later reference in the Hilbert case, we record this in diagrammatic fashion:

Lemma 3.19. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, then $\gamma^* s = (cz+d)s$, i.e. the following diagram commutes:*

$$\begin{array}{ccccc} \hat{\mathbb{G}}_m \times \mathcal{T}(1) & \xrightarrow{\mathrm{m}} & \mathcal{T}(1) & \xleftarrow{\gamma^{-1}} & \mathcal{T}(1) \\ & \nwarrow^{(cz+d)\times s} & \uparrow{\scriptstyle \gamma^* s} & & \uparrow{\scriptstyle s} \\ & & \hat{\mathbb{G}}_a & \xrightarrow{\gamma} & \hat{\mathbb{G}}_a. \end{array}$$

Proof. We first note that the equivariant action of $\Gamma_0(p)$ on $\mathcal{O}(1)$ which is compatible with (2.1) is by letting γ act via $\det(\gamma)^{-1}\gamma$. In particular, γ^{-1} acts as $\gamma^\vee$. The natural fibre action of γ on s is by $\gamma^* s = \gamma^{-1} \circ s \circ \gamma$, i.e. the right square commutes by definition. We therefore have

$$\gamma^* s(z) = \gamma^\vee \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d \\ -c \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} d \\ -c \end{smallmatrix}\right) + c\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} cz+d \\ 0 \end{smallmatrix}\right) = (cz+d)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \bmod \langle \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) \rangle, \tag{3.3}$$

which shows that $\gamma^* s = (cz+d)s$ as desired. □

Definition 3.20. Let $\mathfrak{s} := \pi_{\mathrm{HT}}^*(s)$ be the pullback of s to a section of $\pi_{\mathrm{HT}}^*\mathcal{O}(1) = q^*\omega$.

Since the isomorphism $\pi_{\mathrm{HT}}^*\mathcal{O}(1)$ is equivariant for the $\Gamma_0(p)$-action, the action of $\Gamma_0(p)$ on $\mathfrak{s}$ is

$$\gamma^* \mathfrak{s} = (c\mathfrak{z} + d)\mathfrak{s}, \tag{3.4}$$

where we recall $\mathfrak{z}$ is the pullback of the parameter z to a function on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. We have the following consequence of Lem. 3.17.

Proposition 3.21. *Let x be a (C, C^+)-point of $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ corresponding to a triple $(E, \mu, \alpha : \mathbb{Z}_p^2 \xrightarrow{\sim} T_p E)$. Then via the isomorphism $\pi_{\mathrm{HT}}^*\mathcal{O}(1) = \omega_{\Gamma(p^\infty)}$, we have*

$$\mathfrak{s}(x) = \mathrm{HT}(\alpha(e_1)) \in \omega_E.$$

Proposition 3.22. *Let $\kappa : x \mapsto x^k$. Then there is a natural isomorphism $\omega^\kappa \cong \omega^{\otimes k}$.*

Proof. As $\mathfrak{s}$ is a non-vanishing section of $q^*\omega$ over $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, the sections of $q^*\omega^{\otimes k}$ are all of the form $f \cdot \mathfrak{s}^{\otimes k}$ for $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a)$. Of these sections, the ones coming from sections of $\omega^{\otimes k}$ – that is, those defined over $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$ – are exactly the $\Gamma_0(p)$-equivariant ones. But

$$\gamma^*(f \cdot \mathfrak{s}^{\otimes k}) = \gamma^* f \cdot \gamma^* \mathfrak{s}^{\otimes k} \overset{(3.4)}{=\!=} \gamma^* f \cdot (c\mathfrak{z} + d)^k \mathfrak{s}^{\otimes k} \quad \text{for all } \gamma \in \Gamma_0(p).$$

The $\Gamma_0(p)$-equivariance of $\gamma^*(f \cdot \mathfrak{s}^{\otimes k})$ is thus equivalent to $\gamma^* f = (c\mathfrak{z} + d)^{-k} f = \kappa^{-1}(c\mathfrak{z} + d) f$. □

Remark 3.23. While the analogy to the complex situation is very close, one notable difference is that on the complex upper half plane $\mathcal{H}$ the canonical differential η_{can} satisfies $\gamma^*\eta_{\mathrm{can}} = (cz+d)^{-1}\eta_{\mathrm{can}}$, whereas on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ one has $\gamma^*\mathfrak{s} = (c\mathfrak{z} + d)\mathfrak{s}$. The different signs can be explained as follows. Both constructions of modular forms depend on a canonical trivialisation of the automorphic bundle ω on the covering space, which is $\mathcal{H}$ in the complex case and $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ in the p-adic case. But there is a sign difference in the canonical trivialisation: consider the universal trivialisation $\alpha : \mathbb{Z}^2 \xrightarrow{\sim} H_1(E, \mathbb{Z})$ on $\mathcal{H}$ and let α_i denote the image of the standard basis vector e_i of $\mathbb{Z}^2$. Then the canonical non-vanishing differential η_{can} is defined to be the unique differential such that $\int_{\alpha_1} \eta_{\mathrm{can}} = 1$ under the pairing

$$\int : H_1(E, \mathbb{Z}) \to \omega_E^\vee, \quad \alpha \mapsto \left(w \mapsto \int_\alpha w\right).$$

As a consequence, when we denote by $q^*\omega$ the pullback of ω via $\mathcal{H} \to Y = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$, then using the natural period map $\iota : \mathcal{H} \hookrightarrow \mathbb{P}^1$, $\alpha \mapsto \left(\int_{\alpha_2} w_{\mathrm{can}} : \int_{\alpha_1} w_{\mathrm{can}}\right)$ we have a natural isomorphism $q^*\omega \cong \iota^*\mathcal{O}(-1)$. On the other hand, in the p-adic case, the trivialisation is given using the image α_1 of e_1 under the Hodge–Tate map $\alpha : \mathbb{Z}_p^2 \to T_p E$. The canonical differential is then the image of α_1 under $\mathrm{HT} : T_p E = H_1^{\text{ét}}(E, \mathbb{Z}_p) \to \omega_{E^\vee} = \omega_E$ and as discussed earlier, the period map $\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathbb{P}^1$ therefore induces an isomorphism $q^*\omega_E \cong \pi_{\mathrm{HT}}^*\mathcal{O}(1)$.

In summary, in the complex case one trivialises $\omega_E^\vee$ whereas in the p-adic case it is $\omega_{E^\vee}$, therefore one description of ω_E is by comparing to $\mathcal{O}(-1)$ whereas the other uses $\mathcal{O}(1)$. It is this difference that ultimately leads to the different signs in the definition of modular forms.

3.4. **Comparison to Katz' convergent modular forms.** For $\epsilon = 0$, it has long been known how to construct sheaves of p-adic modular forms, going back to [Kat73, §4]. We briefly present the construction here in the adic language, and sketch how it compares to our setting.

Let us for simplicity assume $\mathcal{U} = \mathrm{Spa}(L)$; the discussion applies without changes for general $\mathcal{U}$. Let $\mathcal{X}^*_{\mathrm{Ig}(p^n)}(0) \to \mathcal{X}^*(0)$ be the n-th Igusa curve, i.e. the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor parametrising trivialisations $\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} H_n^\vee$ of the canonical subgroup. Since this has a natural finite étale formal model, we can form the inverse limit $\mathcal{X}^*_{\mathrm{Ig}(p^\infty)}(0) \to \mathcal{X}^*(0)$ over n as a sousperfectoid space. This is a pro-étale $\mathbb{Z}_p^\times$-torsor known as the Igusa tower, relatively parametrising isomorphisms $\mathbb{Z}_p \xrightarrow{\sim} T_p H^\vee$. As we will see in more detail in the next section, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p)}(0)_a \\ \downarrow \mathfrak{t} & & \downarrow \\ \mathcal{X}^*_{\mathrm{Ig}(p^\infty)}(0) & \longrightarrow & \mathcal{X}^*(0) \end{array}$$

where $\mathfrak{t}$ is given by using the canonical isomorphism $H_n^\vee = E[p^n]/H_n$ and sending a trivialisation $\alpha : \mathbb{Z}_p^2 \xrightarrow{\sim} T_p E$ to

$$\mathbb{Z}_p \xrightarrow{(1,0)} \mathbb{Z}_p^2 \xrightarrow{\sim} T_p E \xrightarrow{\alpha} T_p H^\vee.$$

One now observes that in the case of $\epsilon = 0$, the function $c\mathfrak{z} + d$ is of the form

$$c\mathfrak{z} + d : \mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)}(0) \to \mathbb{Z}_p^\times,$$

where as usual we consider $\mathbb{Z}_p^\times$ as a profinite adic space. One now checks that $\mathfrak{t}$ is equivariant with respect to the map sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto c\mathfrak{z} + d$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_0(p) \times \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \\ \downarrow (c\mathfrak{z}+d)\times\mathfrak{t} & & \downarrow \mathfrak{t} \\ \mathbb{Z}_p^\times \times \mathcal{X}^*_{\mathrm{Ig}(p^\infty)}(0) & \longrightarrow & \mathcal{X}^*_{\mathrm{Ig}(p^\infty)}(0). \end{array}$$

It follows formally (e.g. [Heu19, Lemma 2.8.4]) that ω^κ is the pullback of the pro-étale line bundle on $\mathcal{X}^*(0)$ associated to the cocycle

$$\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{U})^\times.$$

Returning to general $\epsilon \geq 0$ and $\mathcal{U}$, we now use this to give a new proof that ω^κ is analytic:

Proposition 3.24. *For any $0 \leq \epsilon < \epsilon_\kappa^{\mathrm{def}}$, the sheaf ω^κ is an analytic line bundle on $\mathcal{X}^*_{\mathcal{U}}(\epsilon)$.*

Proof. Away from the boundary, it is clear that ω^κ is a pro-étale line bundle, i.e. an invertible module over the completed structure sheaf of $\mathcal{X}_{\mathcal{U}}(\epsilon)_{\mathrm{pro\acute{e}t}}$, as it is defined via a descent datum for the pro-étale torsor $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$. The crucial point is now that by [Heub, Cor. 3.5], such a pro-étale line bundle is already an analytic line bundle if it is analytic on any Zariski-dense open subspace of $\mathcal{X}_{\mathcal{U}}(\epsilon)$. We can thus reduce to proving the statement over the ordinary locus, including the boundary; the analyticity will then automatically overconverge.

We now use that the Igusa tower admits a formal model which is still a pro-étale $\mathbb{Z}_p^\times$-torsor. It follows from [Heub, Prop. 3.8] that ω^κ on $\mathcal{X}^*(0)$ is locally trivial in the analytic topology. □

Remark 3.25. Alternatively, one could use the analyticity criterion [Heub, Cor. 3.6], which says that a pro-étale line bundle on $\mathcal{X}^*(\epsilon) \times \mathcal{U}$ is analytic if it is analytic in each fibre of a Zariski-dense subset of points in each factor. It is clear that ω^κ becomes trivial over the fibre of any $x \in \mathcal{X}^*(\epsilon)(C)$ because the torsor $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon) \to \mathcal{X}^*(\epsilon)$ becomes split over x. On the other hand, Prop. 3.22 says that ω^κ is analytic over the Zariski-dense set of classical points of $\mathcal{W}$.

4. Comparison with Andreatta–Iovita–Pilloni's modular forms

In this section, we prove that the sheaves of p-adic modular forms defined above are canonically isomorphic to those defined by Pilloni in [Pil13] and Andreatta–Iovita–Pilloni in [AIP18]. In order to distinguish their construction from ours, we shall denote the latter sheaf by $\omega^\kappa_{\mathrm{AIP}}$.

This extends the comparison for classical weights proved in Prop. 3.22 above. We first briefly summarise the construction.

4.1. **The Pilloni-torsor.** The construction of $\omega^{\kappa}_{\mathrm{AIP}}$ relies on the Pilloni-torsor $\mathcal{F}_m(\epsilon)$. In this subsection, we will recall its definition, and show how the section $\mathfrak{s}$ from Defn 3.20 induces a natural map $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{F}_m(\epsilon)$ into the Pilloni-torsor, allowing a direct comparison of the modular forms in [AIP18] with those defined in Defn. 3.15 above. This is very similar to [CHJ17, §2.7], which also relies on the section $\mathfrak{s}$, but the present setup makes the comparison slightly easier: it allows for a global comparison map from the perfectoid modular curve to the Pilloni torsor, which makes it possible to avoid auxiliary choices.

We shall discuss the Pilloni-torsor in an analytic setting, rather than dealing with normal formal schemes like in [AIP18]. This is mainly to avoid discussing normalisations in our non-Noetherian setting over $\mathcal{O}_L$ – while this is still possible, it would require more work.

Definition 4.1. For any $m \in \mathbb{Z}_{\geq 1}$, let $0 \leq \epsilon \leq \epsilon_m^{\mathrm{can}} := 1/p^{m+1}$. Like in [Sch15, Definition III.2.12], one can define a canonical formal model $\mathfrak{X}^*(\epsilon)$ of $\mathcal{X}^*(\epsilon)$ with a semi-abelian formal scheme $\mathfrak{E} \to \mathfrak{X}^*(\epsilon)$. By [AIP18, Cor. A.2], this admits a canonical subgroup $\mathfrak{H}_m \subseteq \mathfrak{E}$ of rank p^m characterised by the property that its reduction mod $p^{1-\epsilon}$ equals $\ker F^m$ where F is the relative Frobenius. Let $H_m \subseteq \mathcal{E} \to \mathcal{X}^*(\epsilon)$ be the adic generic fibre, this is the universal canonical subgroup of the semi-abelian adic space over $\mathcal{X}^*(\epsilon)$ (cf the discussion in §2.1).

The Igusa curve $\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon) \to \mathcal{X}^*(\epsilon)$ is now the finite étale $(\mathbb{Z}/p^m\mathbb{Z})^\times$-torsor which relatively represents isomorphisms of group schemes $\mathbb{Z}/p^m\mathbb{Z} \to H_m^\vee$.

Consider the pullback $\omega_{\mathrm{Ig}(p^m)}$ of the conormal sheaf ω to $\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$, and denote the total space of $\omega_{\mathrm{Ig}(p^m)}$ by $\mathcal{T}_m(\epsilon) \to \mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$. Following [Pil13], the Pilloni-torsor is now a certain open subspace $\mathcal{F}_m(\epsilon) \subseteq \mathcal{T}_m(\epsilon)$. We recall this in an analytic setting.

Definition 4.2. For any formal scheme $\mathfrak{Y}$, the generic fibre $\mathcal{Y} = \mathfrak{Y}^{\mathrm{ad}}_\eta$ comes equipped with a morphism of locally ringed spaces $s : (\mathcal{Y}, \mathcal{O}^+_{\mathcal{Y}}) \to \mathfrak{Y}$. For any coherent sheaf $\mathfrak{G}$ on $\mathfrak{Y}$ with generic fibre $\mathcal{G}$, this gives rise to an integral $\mathcal{O}^+_{\mathcal{Y}}$-module $\mathcal{G}^+ := s^*\mathfrak{G}$ on $\mathcal{Y}$.

Applying this to the conormal sheaf of $\mathfrak{E} \to \mathfrak{X}^*(\epsilon)$ and pulling back to $\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$, we see that the sheaf $\omega_{\mathrm{Ig}(p^m)}$ carries a natural integral structure $\omega^+_{\mathrm{Ig}(p^m)} \subseteq \omega_{\mathrm{Ig}(p^m)}$. The same construction applied to the conormal sheaf of the canonical subgroup $\mathfrak{H}_m \subseteq \mathfrak{E} \to \mathfrak{X}^*(\epsilon)$ gives a morphism of $\mathcal{O}^+_{\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)}$-modules $\pi : \omega^+_{\mathrm{Ig}(p^m)} \to \omega^+_{H_m}$. Let Hdg be the Hodge ideal on $\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$ [AIP18, §3.1].

Lemma 4.3. *The following sequence of $\mathcal{O}^+_{\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)}$-modules is exact:*

$$0 \to p^m \mathrm{Hdg}^{-\frac{p^m-1}{p-1}} \omega^+_{\mathrm{Ig}(p^m)} \longrightarrow \omega^+_{\mathrm{Ig}(p^m)} \xrightarrow{\pi} \omega^+_{H_m} \to 0.$$

Proof. We have an exact sequence $0 \to \omega_{\mathfrak{E}[p^m]/\mathfrak{H}_m} \to \omega_{\mathfrak{E}[p^m]} \to \omega_{\mathfrak{H}_m} \to 0$ over $\mathfrak{X}^*(\epsilon)$. The middle term is $\omega_{\mathfrak{E}}/p^m$, whilst the first term has annihilator $\mathrm{Hdg}^{(p^m-1)/(p-1)}$ by [AIP18, Cor. A.4.2]. We thus have an isomorphism $\omega_{\mathfrak{E}}/p^m \mathrm{Hdg}^{-(p^m-1)/(p-1)} \cong \omega_{\mathfrak{H}_m}$. Pulling this back under the morphism of ringed spaces $(\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon), \mathcal{O}^+_{\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)}) \to (\mathcal{X}^*(\epsilon), \mathcal{O}^+_{\mathcal{X}^*(\epsilon)}) \to \mathfrak{X}^*(\epsilon)$ gives the result. □

When we now regard $H_m^\vee$ as a sheaf of sections over $\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$, we have a morphism of sheaves

$$\psi : \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\alpha} H_m^\vee \xrightarrow{\mathrm{HT}} \omega^+_{H_m} \tag{4.1}$$

that defines a canonical section $\psi(1) \in \omega^+_{H_m}$.

Definition 4.4. Let $0 \leq \epsilon \leq \epsilon_m^{\mathrm{can}}$. The *Pilloni-torsor* $\mathfrak{F}_m(\epsilon)$ is the $\mathcal{O}^+_{\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)}$-module defined by

$$\mathfrak{F}_m(\epsilon) := \{r \in \omega^+_{\mathrm{Ig}(p^m)} \mid \pi(r) = \psi(1)\}.$$

Let $\mathcal{F}_m(\epsilon) \hookrightarrow \mathcal{T}_m(\epsilon) \to \mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$ be its total space. By Lem. 4.3, this is an analytic torsor under the group $(1 + p^m \mathrm{Hdg}^{-(p^m-1)/(p-1)} \hat{\mathbb{G}}_a) \to \mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon)$ and an étale torsor over $\mathcal{X}^*(\epsilon)$ for the group $B_m := \mathbb{Z}_p^\times (1 + p^m \mathrm{Hdg}^{-(p^m-1)/(p-1)} \hat{\mathbb{G}}_a)$ when combined with $\mathcal{X}^*_{\mathrm{Ig}(p^m)}(\epsilon) \to \mathcal{X}^*(\epsilon)$.

Definition 4.5. (1) For the universal weight $\kappa^{\mathrm{un}} : \mathcal{W} \to \mathcal{W}$, let T be the weight space parameter given by the function $\kappa^{\mathrm{un}}(q) - 1$ on $\mathcal{W}$ for a fixed topological generator $q \in \mathbb{Z}_p^\times$.

(2) For any $k \in \mathbb{Z}_{\geq 1}$, we define the annulus $\mathcal{W}_k := \mathcal{W}(|T|^{p^k} \leq |p| \leq |T|^{p^{k-1}})$. For $k = 0$, we simply take the disc $\mathcal{W}_0 := \mathcal{W}(|T| \leq |p|)$. Then $\mathcal{W} = \cup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{W}_k$.

(3) Let now $\kappa : \mathcal{U} \to \mathcal{W}$ be any bounded smooth weight, which we may regard as a function $\kappa : \mathbb{Z}_p^\times \times \mathcal{U} \to \hat{\mathbb{G}}_m$. We let $|T_\kappa| := \sup_{(x,u) \in \mathbb{Z}_p^\times \times \mathcal{U}} |\kappa(x,u) - 1|$ and $|\delta_\kappa| := \max(|p|, |T_\kappa|)$. Let $r := 3$ if $p > 2$ and $r := 5$ if $p = 2$. Then we let $0 < \epsilon_\kappa$ be implicitly defined by $|p|^{\epsilon_\kappa} = |\delta_\kappa|^{1/p^{r+1}}$. Finally, for any $k \in \mathbb{Z}_{\geq 0}$, we let $\mathcal{U}_k := \kappa^{-1}(\mathcal{W}_k)$; then $\mathcal{U} = \cup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{U}_k$.

Remark 4.6. One checks easily that $\epsilon_\kappa \leq \epsilon_\kappa^{\mathrm{def}}$. Moreover, we note that by definition, $\epsilon_\kappa^{\mathrm{def}}$ is such that we can define our sheaf of integral modular forms on $\mathcal{X}(\epsilon)$ for $0 \leq \epsilon < \epsilon_\kappa^{\mathrm{def}}$, while for $0 \leq \epsilon < \epsilon_\kappa$ this sheaf will be invertible (although we do not claim this is the optimal bound).

The sheaf $\omega_{\mathrm{AIP}}^\kappa$ is then defined in [Pil13] and [AIP18] as follows.

Definition 4.7. Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight. In order to define $\omega_{\mathrm{AIP}}^\kappa$, we need to split this up into opens $\mathcal{U}_k$ as in Defn. 4.5: Let $0 \leq \epsilon < \epsilon_\kappa$. Fix now any $k \in \mathbb{Z}_{\geq 0}$ and set $m = r + k - 1$ (this implies $\epsilon_\kappa \leq \epsilon_m^{\mathrm{can}}$ and $\epsilon_\kappa \leq \epsilon_\kappa^{\mathrm{def}}$).

(1) Using the projection $\mathrm{pr} : \mathcal{F}_m(\epsilon) \times \mathcal{U} \to \mathcal{X}_{\mathcal{U}}^*(\epsilon)$, we define

$$\omega_{\mathrm{AIP}}^\kappa|_{\mathcal{U}_k}(V) := \{f \in \mathrm{pr}_* \mathcal{O}_{\mathcal{F}_{m(\epsilon)} \times \mathcal{U}}(V) | \gamma^* f = \kappa^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times (1 + p^m \hat{\mathbb{G}}_a) \times \mathcal{U}\}.$$

Prop. 7.10 below shows that these sheaves for different $k \in \mathbb{Z}_{\geq 1}$ can then be glued over $\mathcal{U}$ to get a sheaf $\omega_{\mathrm{AIP}}^\kappa$. We similarly define $\omega_{\mathrm{AIP}}^{\kappa,+}$ by using $\mathcal{O}^+_{\mathcal{F}_{m(\epsilon)} \times \mathcal{U}}$ instead.

(2) For any $n \in \mathbb{Z}_{\geq 1}$ we set $\omega_{\mathrm{AIP},n}^{\kappa,+} := \mathrm{AL}^{n*} \omega_{\mathrm{AIP}}^{\kappa,+}$ where $\mathrm{AL}^n : \mathcal{X}_{\Gamma_0(p^n)}^*(p^n\epsilon)_a \xrightarrow{\sim} \mathcal{X}^*(\epsilon)$ is the Atkin–Lehner isomorphism (corresponding to the matrix $\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$). Let $i : \mathcal{X}_{\Gamma_0(p^n)}^*(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}^*(p^n\epsilon)_a$ be the restriction map. Then by [AIP18, Théorème 6.2.4], there is a canonical isomorphism

$$i^* \omega_{\mathrm{AIP},n}^{\kappa,+} = q_n^* \omega_{\mathrm{AIP}}^{\kappa,+},$$

where $q_n : \mathcal{X}_{\Gamma_0(p^n)}^*(\epsilon)_a \to \mathcal{X}^*(\epsilon)$ is the forgetful map. For $n = 0$ we let $\omega_{\mathrm{AIP},0}^{\kappa,+} := \omega_{\mathrm{AIP}}^{\kappa,+}$.

(3) We therefore set $\omega_{\mathrm{AIP},\infty}^{\kappa,+} := q^* \omega_{\mathrm{AIP}}^{\kappa,+}$ where $q : \mathcal{X}_{\Gamma_0(p^\infty)}^*(\epsilon)_a \to \mathcal{X}^*(\epsilon)$ is the forgetful map.

4.2. **The comparison morphism.** The following is the main result of this section.

Theorem 4.8. *Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight, let $0 \leq \epsilon \leq \epsilon_\kappa$, and let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then there is a natural isomorphism of $\mathcal{O}^+$-modules on $\mathcal{X}_{\mathcal{U},\Gamma_0(p^n)}^*(\epsilon)_a$*

$$\omega_n^{\kappa,+} \xrightarrow{\sim} \omega_{\mathrm{AIP},n}^{\kappa,+}.$$

In particular, the $\omega_n^{\kappa,+}$ are invertible $\mathcal{O}^+$-modules. By inverting p, we also obtain isomorphisms of invertible $\mathcal{O}$-modules $\omega_n^\kappa \xrightarrow{\sim} \omega_{\mathrm{AIP},n}^\kappa$ on $\mathcal{X}_{\mathcal{U},\Gamma_0(p^n)}^*(\epsilon)_a$. Moreover, this induces a Hecke equivariant isomorphism between the respective spaces of modular forms.

For $n = 0$, this in particular gives a canonical isomorphism $\omega^{\kappa,+} \cong \omega_{\mathrm{AIP}}^{\kappa,+}$ on $\mathcal{X}_{\mathcal{U}}^*(\epsilon)$.

The proof of Thm. 4.8 now relies on the following lemma (cf [AIP18, p 31]):

Lemma 4.9. *Let $\epsilon \leq \epsilon_m^{\mathrm{can}}$. Then $\mathfrak{s}$ induces a morphism over $\mathcal{X}^*(\epsilon)$*

$$\mathfrak{s} : \mathcal{X}_{\Gamma(p^\infty)}^*(\epsilon)_a \to \mathcal{F}_m(\epsilon).$$

Let $\tilde{\mathfrak{s}} := \mathfrak{s} \circ u_n$ where $u_n := \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^n)$, we have $\gamma^* \tilde{\mathfrak{s}} = (c\mathfrak{z} + d)\tilde{\mathfrak{s}}$, that is the diagram

$$\begin{array}{ccccc} \mathcal{X}_{\Gamma(p^\infty)}^*(p^n\epsilon)_a & \xrightarrow{\gamma} & \mathcal{X}_{\Gamma(p^\infty)}^*(p^n\epsilon)_a & \longrightarrow & \mathcal{X}_{\Gamma_0(p^n)}^*(p^n\epsilon)_a \\ \downarrow{\scriptstyle (c\mathfrak{z}+d)\times\tilde{\mathfrak{s}}} & & \downarrow{\scriptstyle \tilde{\mathfrak{s}}} & & \downarrow{\scriptstyle \mathrm{AL}^n} \\ B_m \times \mathcal{F}_m(\epsilon) & \xrightarrow{\mathrm{m}} & \mathcal{F}_m(\epsilon) & \longrightarrow & \mathcal{X}^*(\epsilon), \end{array} \tag{4.2}$$

commutes, where m *denotes the respective action maps.*

Remark 4.10. In fact, the map $\mathfrak{s}$ factors through the forgetful map $\mathcal{X}_{\Gamma(p^\infty)}^*(\epsilon)_a \to \mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a$.

Proof. Prop. 7.11 will prove a more general version of the first part away from the cusps. The morphism extends since the relative moduli interpretation of $\mathcal{F}_m(\epsilon) \to \mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon) \to \mathcal{X}(\epsilon)$ also holds over the cusps.

A more general version of the second part will be proved in Lem. 7.12 below, we now explain the argument in the elliptic case: Recall that $\mathfrak{s} = \pi_{\mathrm{HT}}^* s$. By $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariance of π_{HT}, we therefore have $\tilde{\mathfrak{s}} = \pi_{\mathrm{HT}}^* \tilde{s}$ for the section $\tilde{s} := u_n^* s$ on $\hat{\mathbb{G}}_a \subseteq \mathbb{P}^1$. It therefore suffices to prove that $\gamma^* \tilde{s} = (cz + d)\tilde{s}$. For this we first note that the action of u_n sends $(z : 1)$ to $(p^n z : 1)$, and therefore

$$u_n^* z = p^n z \tag{4.3}$$

In particular, u_n sends $\hat{\mathbb{G}}_a \subseteq \mathbb{P}^1$ onto $p^n \hat{\mathbb{G}}_a \subseteq \mathbb{P}^1$. Note that this is preserved by the action of $\Gamma^0(p^n) := \{\left(\begin{smallmatrix} * & b \\ * & * \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathbb{Z}_p) | b \in p^n \mathbb{Z}_p\}$. By the same computation as in (3.3), we see that on $p^n \hat{\mathbb{G}}_a$, for any $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma^0(p^n)$, we have $\gamma'^* s = (c'z + d')s$. For $\gamma' := u_n \gamma u_n^{-1}$, this shows

$$\gamma^* \tilde{s} = \gamma^* u_n^* s = u_n^* (u_n \gamma u_n^{-1})^* s = u_n^* \left(\begin{smallmatrix} a & p^n b \\ p^{-n} c & d \end{smallmatrix}\right)^* s = u_n^* ((p^{-n} cz + d)s) \overset{(4.3)}{=} (cz + d)\tilde{s}. \qquad \square$$

Proof of Thm. 4.8. It suffices to prove this locally on $\mathcal{W}$, so we may assume that κ has image in $\mathcal{W}_k$ for some $k \in \mathbb{Z}_{\geq 0}$. Set $m = r + k - 1$. We start with the case of $n \in \mathbb{Z}_{\geq 1}$.

Let f be a section of $\omega_{\mathrm{AIP}}^{\kappa,+}$. For simplicity of notation, let us assume that f is a global section, even though the proof works for any section. We may then regard f as a map $\mathcal{F}_m(\epsilon) \times \mathcal{U} \to \hat{\mathbb{G}}_a$. To see that $\tilde{\mathfrak{s}}^* f = f \circ \tilde{\mathfrak{s}}$ is a section of $\omega_n^{\kappa,+}$, we use that for any $\gamma \in \Gamma_0(p^n)$, the diagram

$$\begin{array}{ccc}
\mathcal{X}^*_{\Gamma(p^\infty)}(p^n\epsilon)_a \times \mathcal{U} & \xrightarrow{\gamma} & \mathcal{X}^*_{\Gamma(p^\infty)}(p^n\epsilon)_a \times \mathcal{U} \\
\downarrow{\scriptstyle (c\mathfrak{z}+d)\times\tilde{\mathfrak{s}}\times\mathrm{id}} & & \downarrow{\scriptstyle \tilde{\mathfrak{s}}\times\mathrm{id}} \\
\mathbb{Z}_p^\times(1 + p^m\hat{\mathbb{G}}_a) \times \mathcal{F}_m(\epsilon) \times \mathcal{U} & \xrightarrow{\mathrm{m}\times\mathrm{id}} & \mathcal{F}_m(\epsilon) \times \mathcal{U} \\
\downarrow{\scriptstyle \kappa^{-1}\times f} & & \downarrow{\scriptstyle f} \\
\hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_a & \xrightarrow{\mathrm{m}} & \hat{\mathbb{G}}_a
\end{array}$$

commutes, where m denotes the multiplication map. Here the top square commutes by Lem. 4.9 and the bottom square is commutative by definition of $\omega_{\mathrm{AIP}}^{\kappa,+}$. The outer square now shows that

$$\gamma^*(\tilde{\mathfrak{s}}^* f) = \kappa^{-1}(c\mathfrak{z} + d)\tilde{\mathfrak{s}}^* f$$

as desired. This gives a natural morphism of $\mathcal{O}^+_{\mathcal{X}^*_{\mathcal{U}}(\epsilon)}$-modules $\omega_{\mathrm{AIP}}^{\kappa,+} \to \mathrm{AL}_*^n \omega_n^{\kappa,+}$ which in turn induces $\tilde{\mathfrak{s}}^* : \omega_{\mathrm{AIP},n}^{\kappa,+} = \mathrm{AL}^{n*} \omega_{\mathrm{AIP}}^{\kappa,+} \to \omega_n^\kappa$.

To see this is an isomorphism, recall that $\omega_{\mathrm{AIP}}^{\kappa,+}$ is invertible, so locally on some open $U \subseteq \mathcal{X}_{\mathcal{U}}(\epsilon)$ we can find f that is invertible as an element of $\mathcal{O}^+_{\mathcal{F}_m(\epsilon)\times\mathcal{U}}(U)$. Thus $\omega_{\mathrm{AIP}|\mathcal{U}}^{\kappa,+} = f\mathcal{O}_U^+$. Let $V := \mathrm{AL}^{-n}(U)$; then $\tilde{\mathfrak{s}}^* f \in q_* \mathcal{O}^+_{\mathcal{X}^*_{\mathcal{U},\Gamma(p^\infty)(p^n\epsilon)_a}}(V)$ is invertible. It now follows from Prop. 3.8 that we have $\omega^{\kappa,+}|_V = \tilde{\mathfrak{s}}^* f \mathcal{O}_V^+$. Thus $\tilde{\mathfrak{s}}^*$ is an isomorphism locally over U. Since $\mathcal{X}_{\mathcal{U}}(\epsilon)$ can be covered by such U, this completes the proof in the case of $n \in \mathbb{Z}_{\geq 1}$.

The case of $n = 0$ follows from the case of $n = 1$ since we have $\omega^{\kappa,+} = \mathrm{AL}_* \omega_1^{\kappa,+} = \omega_{\mathrm{AIP}}^{\kappa,+}$. For $n = \infty$, the same argument works for the diagram

$$\begin{array}{ccccc}
\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \xrightarrow{\gamma} & \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \\
\downarrow{\scriptstyle d\times\mathfrak{s}} & & \downarrow{\scriptstyle \mathfrak{s}} & & \downarrow{\scriptstyle q} \\
\mathbb{Z}_p^\times \times \mathcal{F}_m(\epsilon) & \xrightarrow{\mathrm{m}} & \mathcal{F}_m(\epsilon) & \longrightarrow & \mathcal{X}^*(\epsilon).
\end{array} \tag{4.4}$$

which induces an isomorphism $\mathfrak{s}^* : q^* \omega_{\mathrm{AIP}}^+ \to \omega_\infty^{\kappa,+}$ as desired. Lastly, the statement about Hecke equivariance is a special case of Prop. 10.8. $\square$

5. Perfectoid Hilbert modular varieties for G^*

For the remainder of the paper, we move on to Hilbert modular forms. In this section, we recall the classical Hilbert modular varieties for G and G^*, and the perfectoid versions for G^*.

5.1. **Classical Hilbert modular varieties for G and G^*.** The content of this section is mainly classical and can be found in many sources, e.g. [TX16, §2] and [Hid04, §4].

Notation 5.1. (1) Let F be a totally real field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$ and absolute different $\mathfrak{d} \subseteq \mathcal{O}_F$. Let Σ denote the set of infinite places of F.
(2) Recall that we fixed a rational prime p. We set $\mathcal{O}_p := \mathcal{O}_F \otimes \mathbb{Z}_p = \oplus_{\mathfrak{p}|p}\mathcal{O}_{F_\mathfrak{p}}$, where $\mathfrak{p}|p$ ranges over the prime ideals of $\mathcal{O}_F$ over p and where $F_\mathfrak{p}$ is the completion of F at $\mathfrak{p}$.
(3) For any fractional ideal $\mathfrak{r}$, let $\mathfrak{r}^+$ denote the totally positive elements in $\mathfrak{r}$; in general '+' will denote 'totally positive'. We have a non-degenerate trace pairing $\mathrm{Tr} : \mathfrak{r} \times \mathfrak{r}^{-1}\mathfrak{d}^{-1} \to \mathbb{Z}$.
(4) Let $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ and let $G^* := G \times_{\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m} \mathbb{G}_m$, where the map $G \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ is given by the determinant morphism and $\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ is given by the diagonal map.

Let $\mathcal{S} := \mathcal{H}^\Sigma$, where $\mathcal{H} \subset \mathbb{C}$ is the standard upper half-plane. For K an open compact subgroup of $G(\mathbb{A}_f)$, there exists a Shimura variety $\mathrm{Sh}_K(G, \mathcal{S})$ of level K over $\mathbb{Q}$. Similarly, if K^* is an open compact subgroup of $G^*(\mathbb{A}_f)$, there exists a Shimura variety $\mathrm{Sh}_{K^*}(G^*, \mathcal{S})$ over $\mathbb{Q}$. These Shimura varieties are the *Hilbert modular varieties*. The Shimura variety $\mathrm{Sh}_{K^*}(G^*, \mathcal{S})$ is of PEL type, therefore of Hodge type, whereas the Shimura variety $\mathrm{Sh}_K(G, \mathcal{S})$ is of abelian type.

Definition 5.2. Let $\mathfrak{c} \subseteq \mathcal{O}_F$ be any nonzero ideal and let N be coprime to $\mathfrak{c}$.
(1) Let $K_\mathfrak{c} := G(\mathbb{A}_f) \cap \begin{pmatrix} \widehat{\mathcal{O}}_F & (\mathfrak{cd})^{-1}\widehat{\mathcal{O}}_F \\ \mathfrak{cd}\widehat{\mathcal{O}}_F & \widehat{\mathcal{O}}_F \end{pmatrix}$. Let $K^*_\mathfrak{c} := G^*(\mathbb{A}_f) \cap K_\mathfrak{c}$.
(2) Let $K_0(\mathfrak{c}, N) := \{\gamma \in K_\mathfrak{c} | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\}$ and $K^*_0(\mathfrak{c}, N) := K_0(\mathfrak{c}, N) \cap K^*_\mathfrak{c}$.
(3) Let $K_1(\mathfrak{c}, N) := \{\gamma \in K_\mathfrak{c} | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}$. Let $K^*_1(\mathfrak{c}, N) := K_1(\mathfrak{c}, N) \cap K^*_\mathfrak{c}$.
(4) Let $K(\mathfrak{c}, N) := \{\gamma \in K_\mathfrak{c} | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}$. Let $K^*(\mathfrak{c}, N) := K(\mathfrak{c}, N) \cap K^*_\mathfrak{c}$.

5.1.1. *Moduli problems for Hilbert–Blumenthal abelian varieties.* Hilbert modular varieties arise as solutions to moduli problems of abelian varieties with level structures.

Definition 5.3. Let S be any scheme. Let $\mathfrak{c} \subseteq \mathcal{O}_F$ be an ideal.
(1) A Hilbert–Blumenthal abelian variety (HBAV) over S is a triple (A, ι, λ) consisting of an abelian variety A over S with real multiplication $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$ and a $\mathfrak{c}$-polarisation $\lambda : A \otimes \mathfrak{c} \xrightarrow{\sim} A^\vee$, such that ι is stable under the Rosati-involution.
(2) Given a HBAV (A, ι, λ), we refer to (A, ι) as the underlying abelian variety with real multiplication (AVRM).
(3) A morphisms of HBAVs is an $\mathcal{O}_F$-linear morphisms $f : A \to A'$ of abelian S-schemes for which $\lambda = f^\vee \circ \lambda' \circ f$.

The Shimura varieties for G^* represent moduli problems given by $\mathfrak{c}$-polarised HBAVs with additional rigidifying level structure, which we shall discuss next. In contrast, the Shimura varieties for G are only coarse moduli spaces parametrising triples $(A, \iota, [\lambda])$ where (A, ι) is an AVRM and $[\lambda] = \{\nu\lambda | \nu \in \mathcal{O}_F^{\times,+}\}$ is a *polarisation class*, plus level structure.

Definition 5.4 (Level structures for G)**.** Let (A, ι, λ) be a $\mathfrak{c}$-polarised HBAV, let $N \in \mathbb{Z}_{\geq 4}$ with $(N, p) = 1$ and let $n \in \mathbb{Z}_{\geq 0}$.
(1) A μ_N-level structure is a closed immersion of $\mathcal{O}_F$-module schemes $\mathfrak{d}^{-1} \otimes_\mathbb{Z} \mu_N \hookrightarrow A[N]$.
(2) A $\Gamma_0(p^n)$-level structure is an $\mathcal{O}_F$-submodule scheme $\Phi_n : C \hookrightarrow A[p^n]$ that is étale locally isomorphic to $\mathcal{O}_F/p^n\mathcal{O}_F$. Via λ, this is equivalent to giving an $\mathcal{O}_F$-submodule scheme
$$C' := \lambda(C \otimes \mathfrak{c}) \hookrightarrow A^\vee[p^n].$$
In the context of Shimura varieties for G^*, we also call the same data a $\Gamma^*_0(p^n)$-level structure, since this level structure appears for both G^* and G.
(3) A $\Gamma_1(p^n)$-level structure is a closed immersion of $\mathcal{O}_F$-module schemes
$$\Phi_n : \mathcal{O}_F/p^n\mathcal{O}_F \hookrightarrow A^\vee[p^n].$$
Again, we also call this a $\Gamma^*_1(p^n)$-level structure,
(4) A $\Gamma(p^n)$-level structure is an isomorphism of $\mathcal{O}_F$-module schemes
$$\alpha_n : (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \xrightarrow{\sim} A^\vee[p^n].$$

Remark 5.5. Our definition of level structures at p is slightly non-standard: usually one would define a $\Gamma(p^n)$-structure to be an isomorphism $(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \xrightarrow{\sim} A[p^n]$. The reason we use the above modified version parametrising $A^\vee[p^n]$ is that the Hodge–Tate morphism is of the form $T_pA^\vee \to \omega_A$, so it is *this* level structure which gives rise at infinite level to canonical sections of ω_A, as required for the definition of modular forms. The isomorphisms $(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \xrightarrow{\sim} A[p^n]$ would instead give sections of $\omega_{A^\vee}$. We note, however, that given a fixed $\mathfrak{c}$-polarisation λ as in the moduli problem for G^*, a $\Gamma(p^n)$-level structure is equivalent to an isomorphism

$$(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \otimes \mathfrak{c}^{-1} \xrightarrow{\alpha_n\otimes\mathfrak{c}^{-1}} A^\vee[p^n] \otimes \mathfrak{c}^{-1} \xrightarrow{\lambda^{-1}} A[p^n].$$

In this case our notion is (non-canonically) isomorphic to the more standard definition.

We now define level structures for G^*. Recall that we have defined $\Gamma_0^* = \Gamma_0$ and $\Gamma_1^* = \Gamma_1$-level structures to be the same, but at full level we need a slightly different definition, analogous to [Rap78, §1.21]. To motivate this, observe that we can see G as a group preserving a pairing up to similitude, whilst G^* is the subgroup that preserves certain rational structures within this. For the Shimura varieties associated to G^*, we therefore need isomorphisms α_n preserving a rational structure. We will now define the $\mathcal{O}_F$-structure in which the rational structure should live, via an $\mathcal{O}_F$-linear version of the Weil pairing.

Definition 5.6. Let (A, ι, λ) be a HBAV. The Weil pairing $e_{p^n} : A[p^n] \times A^\vee[p^n] \to \mu_{p^n}$ satisfies $e_{p^n}(ax, y) = e_{p^n}(x, ay)$ for $a \in \mathcal{O}_F$ (see [Mum70, Section 20]). The Weil pairing can therefore be extended to an $\mathcal{O}_F$-linear version, by using the trace map to write $\mathfrak{d}^{-1} \simeq \mathrm{Hom}(\mathcal{O}_F, \mathbb{Z})$, setting

$$\begin{aligned} \widetilde{\mathbf{e}}_n : A[p^n] \times A^\vee[p^n] &\longrightarrow \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}, \\ x, y &\longmapsto (a \mapsto e_{p^n}(ax, y)) \end{aligned} \tag{5.1}$$

which is $\mathcal{O}_F$-bilinear and perfect. We call $\widetilde{\mathbf{e}}_n$ the *$\mathcal{O}_F$-linearisation* of e_{p^n}. Note $e_{p^n} = \mathrm{Tr} \circ \widetilde{\mathbf{e}}_n$.

We fix a non-degenerate $\mathcal{O}_F$-linear pairing on $(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \otimes \mathfrak{c}^{-1} \times (\mathcal{O}_F/p^n\mathcal{O}_F)^2$ to compare to the Weil pairing. To this end, fix an isomorphism of free $\mathcal{O}_p$-modules of rank 1

$$\beta : \mathfrak{c}^{-1}\mathcal{O}_p \to \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} T_p\mu_{p^\infty} =: \mathfrak{d}^{-1}(1),$$

or equivalently an $\mathcal{O}_p$-module generator $\beta \in \mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} T_p\mu_{p^\infty} = \mathfrak{c}\mathfrak{d}^{-1}(1)$. Then we get a pairing

$$\begin{aligned} \langle -,- \rangle_{\beta,n} : (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \otimes \mathfrak{c}^{-1} \times (\mathcal{O}_F/p^n\mathcal{O}_F)^2 &\xrightarrow{\beta\times\mathrm{id}} (\mathfrak{d}^{-1}/p^n)^2 \otimes_{\mathbb{Z}} \mu_{p^n} \times (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \\ &\xrightarrow{\det} \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}, \end{aligned}$$

where $\det : [(a,b),(c,d)] \mapsto ad - bc$. Given a $\Gamma(p^n)$-level structure α_n, this fits into a diagram

$$\begin{array}{ccccccc} (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \otimes \mathfrak{c}^{-1} & \times & (\mathcal{O}_F/p^n\mathcal{O}_F)^2 & \xrightarrow{\langle -,- \rangle_{\beta,n}} & \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n} & \xrightarrow{\mathrm{Tr}} & \mu_{p^n} \\ \downarrow{\scriptstyle \lambda^{-1}\circ(\alpha_n\otimes\mathrm{id})} & & \downarrow{\scriptstyle \alpha_n} & & \wr\downarrow{\scriptstyle b} & & \vdots \\ A[p^n] & \times & A^\vee[p^n] & \xrightarrow{\widetilde{\mathbf{e}}_n} & \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n} & \xrightarrow{\mathrm{Tr}} & \mu_{p^n}. \end{array} \tag{5.2}$$

The two pairings into $\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}$ will always be similar, that is there always exists some $b \in \mathrm{Aut}(\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}) = (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$ that makes the above diagram commute.

Definition 5.7. A $\Gamma^*(p^n)$-level structure is a $\Gamma(p^n)$-level structure $\alpha_n : (\mathcal{O}_F/p^n)^2 \xrightarrow{\sim} A^\vee[p^n]$ such that the similitude b in (5.2) lies in the subgroup $(\mathbb{Z}/p^n\mathbb{Z})^\times \subseteq (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$. Equivalently, it is an α_n such that after composing both pairings with the trace map – the final horizontal maps of (5.2) – the two pairings into μ_{p^n} remain similar via $b \in \mathrm{Aut}(\mu_{p^n}) = (\mathbb{Z}/p^n\mathbb{Z})^\times$.

Via its natural action on $(\mathcal{O}_F/p^n\mathcal{O}_F)^2$, the group $G^*(\mathbb{Z}_p/p^n\mathbb{Z}) \subseteq \mathrm{GL}_2(\mathcal{O}_F/p^n\mathcal{O}_F)$ acts on $\Gamma^*(p^n)$-level structures by letting γ act as pre-composition with $\gamma^\vee := \det(\gamma)\gamma^{-1}$.

Remark 5.8. We note that the dual ensures that we obtain a left-action. One reason for us to use $\gamma^\vee$ rather than γ^{-1} to define the action is Lem. 8.23 below.

In the limit $n \to \infty$, the pairings $\langle -,- \rangle_{\beta,n}$ are compatible and define a pairing $\langle -,- \rangle_\beta : \mathcal{O}_p^2 \otimes \mathfrak{c}^{-1} \times \mathcal{O}_p^2 \to \mathfrak{d}^{-1}(1)$. A $\Gamma^*(p^\infty)$-structure is a compatible collection of $\Gamma^*(p^n)$-level structures.

Equivalently, this is an isomorphism $\mathcal{O}_p^2 \to T_pA^\vee$ inducing a (rational) similitude of pairings on $\mathcal{O}_p^2 \otimes \mathfrak{c}^{-1} \times \mathcal{O}_p^2 \to T_pA \times T_pA^\vee$ as above.[b]

5.1.2. *Hilbert modular varieties as moduli spaces.* We now recall the moduli problems that the Shimura varieties $\mathrm{Sh}_{K^*}(G^*, \mathcal{S})$ represent. Consider the functor

$$\mathrm{Sch}\,/_{\mathbb{Z}[1/N]} \longrightarrow \mathrm{Set},$$

sending a scheme S to the set of isomorphism classes of (A, ι, λ, μ), where (A, ι, λ) is a $\mathfrak{c}$-polarised HBAV and μ is a μ_N-level structure which we call the tame level. This functor is represented by the Hilbert moduli scheme $X(\mathfrak{c}, \mu_N)_{\mathbb{Z}[1/N]}$ over $\mathbb{Z}[1/N]$. We denote by $X(\mathfrak{c}, \mu_N)_R$ the base change to any $\mathbb{Z}[1/N]$-algebra R. Then there is an isomorphism

$$X(\mathfrak{c}, \mu_N)_{\mathbb{C}} \cong \mathrm{Sh}_{K_1^*(\mathfrak{c},N)}(G^*, \mathcal{S}).$$

Thus $X(\mathfrak{c}, \mu_N)_{\mathbb{Z}[1/N]}$ is a model for $\mathrm{Sh}_{K_1^*(\mathfrak{c},N)}(G^*, \mathcal{S})$ over $\mathbb{Z}[1/N]$. We write $X := X(\mathfrak{c}, \mu_N)_L$

In the case of G, the Shimura variety $\mathrm{Sh}_{K_1(\mathfrak{c},N)}(G, \mathcal{S})$ also has a canonical model $X_G(\mathfrak{c}, \mu_N)_{\mathbb{Z}[1/N]}$, but this is only a coarse moduli space; for an algebraically closed field C, and appropriate K, the C-points of $X_G(\mathfrak{c}, \mu_N)_{\mathbb{C}}$ parametrise isomorphism classes of tuples $(A, \iota, [\lambda], \mu_N)$, where $[\lambda]$ is a polarisation class. We shall write $X_G := X_G(\mathfrak{c}, \mu_N)_L$.

We obtain the analogous results when we now add level structures at p. Let Γ_p^* be one of $\Gamma_0^*(p^n)$, $\Gamma_1^*(p^n)$ or $\Gamma^*(p^n)$, and let K_p^* be the corresponding subgroup $K_0^*(\mathfrak{c}, p^n)$, $K_1^*(\mathfrak{c}, p^n)$ or $K^*(\mathfrak{c}, p^n)$. Consider the functor $\mathrm{Sch}\,/_{\mathbb{Q}} \to \mathrm{Set}$ sending a scheme S to the set of isomorphism classes of $(A, \iota, \lambda, \mu, \alpha)$, where (A, ι, λ, μ) is as above, and α is a Γ_p^*-level structure. This functor is represented by a Hilbert moduli scheme $X(\mathfrak{c}, \mu_N, \Gamma_p^*)_{\mathbb{Q}}$ whose base-change to $\mathbb{C}$ is isomorphic to $\mathrm{Sh}_{K_1^*(\mathfrak{c},N) \cap K_p^*}(G^*, \mathcal{S})$. Here for $\Gamma_p^* = \Gamma^*(p^n)$ we recall that our definition of this level structure depends on our chosen isomorphism β, and the isomorphism is given by transforming our notion of a $\Gamma^*(p^n)$-level structure into the usual notion, by using β, the complex unit root $\exp(2\pi i/p^n)$ and the given polarisation to make $(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \to A^\vee[p^n]$ into an isomorphism $\mathfrak{d}^{-1} \otimes \mu_{p^n} \times \mathcal{O}_F/p^n\mathcal{O}_F \to A[p^n]$.

Again we abbreviate $X_{\Gamma_p^*} := X(\mathfrak{c}, \mu_N, \Gamma_p^*)_L$. By the natural forgetful map between the moduli functors, we can summarise these Hilbert moduli schemes in the tower of moduli schemes

$$X_{\Gamma^*(p^n)} \longrightarrow X_{\Gamma_1^*(p^n)} \longrightarrow X_{\Gamma_0^*(p^n)} \longrightarrow X.$$

Let $X_{\Gamma_p^*}^*$ denote the minimal compactification of $X_{\Gamma_p^*}$ over L. For tame level, we also have the minimal compactification $X_{\mathcal{O}_L}^*$ over $\mathcal{O}_L$. These can all be defined via base-change from $\mathbb{Q}$ or $\mathbb{Z}[1/N]$, respectively.

Finally in this subsection, we pass to p-adic geometric spaces:

Definition 5.9. Let Γ_p^* be a level at p of the form $\Gamma_0^*(p^n), \Gamma_1^*(p^n), \Gamma^*(p^n)$ for some $n \in \mathbb{Z}_{\geq 0}$.

(1) Let $\mathfrak{X}^* \to \mathrm{Spf}(\mathcal{O}_L)$ be the p-adic completion of $X_{\mathcal{O}_L} \to \mathrm{Spec}(\mathcal{O}_L)$.
(2) We denote by $\mathcal{X}_{\Gamma_p^*}$ the adic analytification of $X_{\Gamma_p^*}$. We thus obtain a tower of adic spaces
$$\mathcal{X}_{\Gamma^*(p^n)} \longrightarrow \mathcal{X}_{\Gamma_1^*(p^n)} \longrightarrow \mathcal{X}_{\Gamma_0^*(p^n)} \longrightarrow \mathcal{X}.$$
(3) In the case of tame level, we also have the compactification $\mathfrak{X}^* \to \mathrm{Spf}(\mathcal{O}_L)$ obtained by completion of $X_{\mathcal{O}_L}^*$. Its adic generic fibre coincides with the analytification $\mathcal{X}^*$ of X^*.

Remark 5.10. Our notation suppresses the dependence on the polarisation ideal $\mathfrak{c}$ (and on the tame level). If required, we will make this clear by writing $\mathfrak{X}_{\mathfrak{c},K_p^*}^*$, $\mathcal{X}_{\mathfrak{c},K_p^*}^*$, etc.

5.2. **Hilbert modular varieties for G^* at infinite level.** Next, we recall the perfectoid Hilbert modular varieties for the group G^*, following [Sch15] and [CS17].

Let $\mathcal{X}^*(\epsilon)$ denote the admissible open subset of $\mathcal{X}^*$ defined by $|\widetilde{\mathrm{Ha}}| \geq |p|^\epsilon$, where $\widetilde{\mathrm{Ha}}$ denote any local lift of the (total) Hasse invariant. Now, for $n \in \mathbb{Z}_{\geq 0}$ take $\epsilon \in [0, (p-1)/p^n) \cap \log|L|$. Then by [AIP16a, Section 3.2] we have an integral model $\mathfrak{X}^*(\epsilon)$ of $\mathcal{X}^*(\epsilon)$. Moreover, abelian

[b]The above choices may not seem natural, but we remark that they do not affect our construction for modular forms G, and are just an auxiliary choice in our construction of G^*-forms. We also note that the Shimura varieties are of course completely canonical, it is the moduli interpretation which requires the choice.

schemes parametrised by $\mathcal{X}^*(\epsilon)$ have a canonical subgroup of level n which agrees with the kernel of the n-th iterated power of the Frobenius map modulo $p^{1-\epsilon}$. Following [Sch15], we let $\mathcal{X}^*_{\Gamma_0^*(p^n)}(\epsilon)_a$ denote the anticanonical locus, parametrising subgroups that intersect the canonical subgroup trivially. The forgetful maps then extend to the minimal compactification and give the anticanonical tower

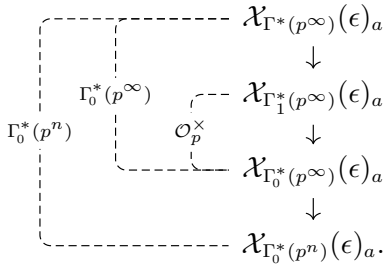
$$\cdots \longrightarrow \mathcal{X}^*_{\Gamma_0^*(p^2)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_0^*(p)}(\epsilon)_a \longrightarrow \mathcal{X}^*(\epsilon).$$

One then has the following analogue of the results in [Sch15, §III]:

Theorem 5.11. *There are perfectoid spaces that are the tilde-limits*

(1) $\mathcal{X}_{\Gamma_0^*(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a$.
(2) $\mathcal{X}_{\Gamma_1^*(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma_1^*(p^n)}(\epsilon)_a$.
(3) $\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma^*(p^n)}(\epsilon)_a$.
(4) $\mathcal{X}_{\Gamma^*(p^\infty)} \sim \varprojlim_n \mathcal{X}_{\Gamma^*(p^n)}$.

They fit into the following tower of pro-étale torsors for the indicated profinite groups



where $\Gamma_0^*(p^n) \subseteq G^*(\mathbb{Z}_p)$ *is the subgroup of matrices that are upper triangular mod* p^n*, and* $\Gamma_0^*(p^\infty) \subseteq G^*(\mathbb{Z}_p)$ *is the subgroup of upper-triangular matrices.*

Part (4) is proved in [Sch15, §Theorem IV.1.1] in the case that L is algebraically closed, and under the additional assumption that the embedding $G^* \hookrightarrow \mathrm{GSp}_{2g}$ sends K^p into a subgroup of $\mathrm{GSp}_{2g}(\hat{\mathbb{Z}})$. In our case, this means that $\mathfrak{c} = 1$. One can deduce the general case from this by first shrinking the level and then quotienting by a Galois action, which by [Han, Theorem 1.4] or [HJ, Theorem 5.8] is again a perfectoid space. Part (3) of the proposition then follows by restriction, and one can modify the argument to also obtain statements (1) and (2).

Alternatively, to prove Theorem 5.11 one can follow Scholze's construction in the Siegel case [Sch15, §III], as we shall now demonstrate: here we note that since we ignore the boundary, we do not need to worry about ramification. One first shows (cf [Sch15, Theorem III.2.15]):

Lemma 5.12. *Division by the canonical subgroup defines a natural map*

$$\phi : \mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$$

that reduces to the relative Frobenius mod $p^{1-\delta}$ *where* $\delta = \frac{p+1}{p}\epsilon$.

Proof. We argue as in [Sch15, Thm. III.2.15]. Let $A \to \mathfrak{X}(p^{-1}\epsilon)$ be the universal abelian scheme. Then the abelian scheme $A' := A/H_1$ defines a morphism $\phi : \mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}$. Locally on any affine open $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ defined over $\mathbb{Z}_p$, this corresponds to a morphism $\phi : R \to R\langle T\rangle/(T\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})$. Since $H_1 \equiv \ker F \bmod p^{1-p^{-1}\epsilon}$, we have $A' \equiv A^{(p)} \bmod p^{1-p^{-1}\epsilon}$. Consequently, ϕ reduces mod $p^{1-p^{-1}\epsilon}$ to the relative Frobenius on R. Here $R^{(p)} = R$ since R is already defined over $\mathbb{Z}_p$. In particular, since $\mathrm{Ha}(A^{(p)}) = \mathrm{Ha}(A)^p$, we have $\phi(\widetilde{\mathrm{Ha}}) \equiv \widetilde{\mathrm{Ha}}^p \bmod p^{1-p^{-1}\epsilon}$. Consequently,

$$T^p\phi(\widetilde{\mathrm{Ha}}) \equiv T^p\widetilde{\mathrm{Ha}}^p = p^\epsilon \bmod p^{1-p^{-1}\epsilon}$$

inside $R\langle T\rangle/(T\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})$. We can therefore find $u \in R$ such that

$$T^p\phi(\widetilde{\mathrm{Ha}}) = p^\epsilon + p^{1-p^{-1}\epsilon}u = p^\epsilon(1 + p^{1-\delta}u).$$

Sending $T \mapsto \phi(T) := T^p(1 + p^{1-\delta}u)^{-1}$ therefore defines a unique extension

$$\phi : R\langle T\rangle/(T\widetilde{\mathrm{Ha}} - p^\epsilon) \to R\langle T\rangle/(T\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})$$

giving the desired lift to a map

$$\phi : \mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon). \tag{5.3}$$

Since $\phi \bmod p^{1-\epsilon}$ is given by the Frobenius on R, and by sending $T \mapsto T^p(1+p^{1-\delta}u)^{-1} \equiv T^p \bmod p^{1-\delta}$, this map reduces to the relative Frobenius mod $p^{1-\delta}$ as desired.

It remains to extend this to the minimal compactification. Since the boundary is already contained in the ordinary locus, it suffices to consider the case of $\epsilon = 0$, and thus $\delta = 0$. It moreover suffices to consider the case of $L = \mathbb{Q}_p^{\mathrm{cyc}}$; the general case follows by base-change.

Since $X^*_{\mathbb{F}_p}$ is normal (see [Cha90, Thm. 4.3] and the discussion in [KL05, 1.8.1]), and the cusps are of codimension ≥ 2, we can now apply Scholze's version of Hartog's extension principle, [Sch15, Lem. III.2.10] to see that ϕ extends uniquely over the boundary to a map

$$\phi : \mathfrak{X}^*(0) \to \mathfrak{X}^*(0)$$

which still reduces to the relative Frobenius mod p. Glueing this to (5.3) proves the lemma. □

Proof of Thm. 5.11. We can argue as in [Sch15, Cor. III.2.19]. For every $n \in \mathbb{N}$ we have by Lem. 5.12 a morphism $\phi : \mathfrak{X}^*(p^{-n-1}\epsilon) \to \mathfrak{X}^*(p^{-n}\epsilon)$ that reduces to the relative Frobenius mod $p^{1-\frac{p+1}{p^{n+1}}\epsilon}$ and in particular mod $p^{1-\delta}$ where $\delta := \frac{p+1}{p}\epsilon$. Consequently, $\mathfrak{X}^*(p^{-\infty}\epsilon) := \varprojlim_\phi \mathfrak{X}^*(p^{-n}\epsilon)$ is a flat formal scheme for which the relative Frobenius mod $p^{1-\delta}$ is an isomorphism. It follows that the generic fibre $\mathcal{X}^*(p^{-\infty}\epsilon)$ is perfectoid and moreover, by [SW13, Prop. 2.4.2], on the generic fibres we have

$$\mathcal{X}^*(p^{-\infty}\epsilon) \sim \varprojlim_\phi \mathcal{X}^*(p^{-n}\epsilon).$$

By [SW13, Prop. 2.4.3], we can now restrict to the open modular curve to deduce that there is a perfectoid tilde-limit $\mathcal{X}(p^{-\infty}\epsilon) \sim \varprojlim_\phi \mathcal{X}(p^{-n}\epsilon)$. Since the Atkin–Lehner isomorphisms AL^n define an isomorphism of inverse systems of the anticanonical tower to the system

$$\cdots \longrightarrow \mathcal{X}(p^{-n-1}\epsilon) \xrightarrow{\phi} \mathcal{X}(p^{-n}\epsilon) \longrightarrow \dots$$

we equivalently have $\mathcal{X}_{\Gamma_0^*(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a$, as desired. This proves part (1).

To deduce parts (2) and (3), we use that the forgetful morphism $\mathcal{X}_{\Gamma_1^*(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a$ is finite étale. By pulling these back from varying n, we obtain a tower of finite étale morphisms

$$\begin{array}{ccccc}
\mathcal{X}_{\Gamma_0^*(p^\infty)}(\epsilon)_a & \longleftarrow & \mathcal{X}_{\Gamma_0^*(p^\infty)\cap\Gamma_1^*(p^n)}(\epsilon)_a & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \\
\mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a & \longleftarrow & \mathcal{X}_{\Gamma_1^*(p^n)}(\epsilon)_a & &
\end{array}$$

Since perfectoid tilde-limits of inverse systems of perfectoid spaces with affinoid transition maps exist, we obtain a perfectoid tilde-limit

$$\mathcal{X}_{\Gamma_1^*(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma_0^*(p^\infty)\cap\Gamma_1^*(p^n)}(\epsilon)_a.$$

This proves part (2). Part (3) follows similarly using that $\mathcal{X}_{\Gamma_1^*(p^n)} \to \mathcal{X}_{\Gamma_0^*(p^n)}$ is finite étale.

Since the morphisms $\mathcal{X}_{\Gamma_1^*(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a$ and $\mathcal{X}_{\Gamma^*(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0^*(p)}(\epsilon)_a$ and $\mathcal{X}_{\Gamma^*(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a$ are finite étale torsors for the groups $(\mathcal{O}_F/p^n\mathcal{O}_F)^\times$, $\{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \mathbb{G}^*(\mathbb{Z}/p^n\mathbb{Z}) | c \in p\mathcal{O}_F/p^n\mathcal{O}_F\}$ and $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathbb{G}^*(\mathbb{Z}/p^n\mathbb{Z})\}$, respectively, the last statement follows from the fact that perfectoid tilde-limits commute with fibre products.

It remains to prove (4), which we deduce from (3) using the $G^*(\mathbb{Q}_p)$-action at infinite level recalled in §8.4.1: like in [Sch15, §III.3], it suffices to prove that on the level of topological spaces we have $G^*(\mathbb{Q}_p)|\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a| = |\mathcal{X}_{\Gamma(p^\infty)}| := |\varprojlim_n \mathcal{X}_{\Gamma(p^n)}|$. But as it suffices to prove this after passing to a smaller K^p and any field extension of C, we can reduce to the case considered in [Sch15, Theorem IV.1.1]. This finishes the proof of the theorem. □

Remark 5.13. As in the elliptic case, we have a moduli description of the (C, C^+)-points of $\mathcal{X}_{\Gamma^*(p^\infty)}$ for any perfectoid extension C of L: They are in functorial one-to-one correspondence with isomorphism classes of tuples $(A, \iota, \lambda, \mu_N, \alpha)$ where (A, ι, λ) is an ϵ-nearly ordinary $\mathfrak{c}$-polarised

HBAV over C with tame level μ_N, together with a $\Gamma^*(p^\infty)$-level structure $\alpha : \mathcal{O}_p^2 \xrightarrow{\sim} T_p A^\vee$. The subspace $\mathcal{X}_{\Gamma_0^*(p)}(\epsilon)_a$ represents those tuples for which $\alpha(1,0)$ generates a subgroup of $A^\vee[p]$ that is different to the canonical subgroup.

Definition 5.14. The $G^*(\mathbb{Z}/p^n\mathbb{Z})$-actions on $\mathcal{X}_{\Gamma^*(p^n)}$ in the limit give rise to a $G^*(\mathbb{Z}_p)$-action on $\mathcal{X}_{\Gamma^*(p^\infty)}$ which in terms of moduli can be described as follows: the action of $\gamma \in G^*(\mathbb{Z}_p) \subseteq \mathrm{GL}_2(\mathcal{O}_p)$ sends any HBAV $(A, \iota, \lambda, \alpha : \mathcal{O}_p^2 \to T_p A^\vee)$ to $(A, \iota, \lambda, \alpha \circ \gamma^\vee)$.

5.3. **The Hodge–Tate period morphism and its image.** For any adic space S over $\mathrm{Spa}(L)$, we denote by $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} S$ the functor on affinoid $(L, \mathcal{O}_L)$-algebras given by $(R, R^+) \mapsto S(R_F, R_F^+)$, where $R_F := R \otimes_{\mathbb{Q}} F$ and R_F^+ is the integral closure of $R^+ \otimes_{\mathbb{Z}} \mathcal{O}_F$ in R_F. If $S = X^{\mathrm{an}}$ is the analytification of a variety X over L for which the usual restriction of scalars $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} X$ is representable by a variety, we have $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} S = (\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} X)^{\mathrm{an}}$. For all spaces we need below, this shows that $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} S$ is representable by an adic space.

For example, $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$ is the adic analytification of the finite type scheme representing the functor that sends any L-algebra R to the set $\mathbb{P}^1(R \otimes_{\mathbb{Z}} \mathcal{O}_F)$. This is the flag variety of G^*. By [CS17, Thm. 2.1.3], there is a Hodge–Tate period map of the form

$$\pi_{\mathrm{HT}} : \mathcal{X}_{\Gamma^*(p^\infty)} \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1.$$

Remark 5.15. On points, this map has the following moduli interpretation: let C/L be a complete algebraically closed field and let A a $\mathfrak{c}$-polarised HBAV over C. Then the Hodge–Tate filtration is a short exact sequence of $C \otimes_{\mathbb{Z}_p} \mathcal{O}_p$-modules

$$0 \longrightarrow \mathrm{Lie}(A^\vee)(1) \longrightarrow T_p A^\vee \otimes_{\mathbb{Z}_p} C \xrightarrow{\mathrm{HT}_A} \omega_A \longrightarrow 0$$

Now, a point $x \in \mathcal{X}_{\Gamma^*(p^\infty)}(C, C^+)$ gives rise to a trivialisation $\mathcal{O}_p^2 \xrightarrow{\sim} T_p A^\vee$ which we can use to consider the above as a filtration of $\mathcal{O}_p^2 \otimes_{\mathbb{Z}_p} C$ of rank 1. This defines the desired point $\pi_{\mathrm{HT}}(x) \in \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1(C, C^+) = \mathbb{P}^1(\mathcal{O}_p \otimes_{\mathbb{Z}_p} C)$.

For the definition of Hilbert modular forms, it will be important for us to bound the image of the anticanonical locus under the Hodge–Tate period map. More precisely, our goal is to compare this to a family of neighbourhoods of $\mathbb{P}^1(\mathcal{O}_p) \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$ which we shall now define.

Definition 5.16. Recall from Defn. 3.11 that we had defined adic groups $\mathbb{G}_a$, $\mathbb{G}_m$, $\hat{\mathbb{G}}_a$, $\hat{\mathbb{G}}_m$.

(1) By applying the functor $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} -$, we obtain adic spaces $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{G}_m$, $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{G}_a$, and open subspaces $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_m$, $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a$.
(2) Given a point $x \in \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{G}_a$, and an element $z \in L$ with $|z| = r$, we shall call the open subspace $B_r(x) := x + z \,\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a \subseteq \mathbb{G}_a$ the ball of radius r around x.

Definition 5.17. The subspace $\mathbb{P}^1(\mathcal{O}_p) = \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1(\mathbb{Z}_p)$ is a profinite set, and therefore has a geometric incarnation as a morphism $\mathbb{P}^1(\mathcal{O}_p) \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$, where as usual we also write $\mathbb{P}^1(\mathcal{O}_p)$ for the associated profinite perfectoid space.

We embedded $\mathbb{G}_a \hookrightarrow \mathbb{P}^1$ via $z \mapsto (z : 1)$. By applying $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}$, this defines an open subspace $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{G}_a \hookrightarrow \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$. We also have $\mathcal{O}_p =: B_0(\mathcal{O}_p : 1) \hookrightarrow \mathbb{P}^1(\mathcal{O}_p)$ via $a \mapsto (a : 1)$ for $a \in \mathcal{O}_p$. For $r \in (0, 1] \cap |L|$, we define the open neighbourhood $B_r(\mathcal{O}_p : 1) \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$ of $B_0(\mathcal{O}_p : 1)$ to be the union of all balls of radius r around points in $\mathcal{O}_p \hookrightarrow \hat{\mathbb{G}}_a \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$. We make analogous definitions for open subspaces $B_r(\mathcal{O}_p^\times : 1)$ and $B_r(1 : p\mathcal{O}_p)$ of $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$.

Proposition 5.18. *Let $1 > r \geq 0$. Then for any $m \in \mathbb{Z}_{\geq 1}$ with $1/p^m \leq r$ and any $0 \leq \epsilon \leq 1/2p^m$, or $\epsilon \leq 1/3p^m$ if $p = 3$, or $\epsilon \leq 1/4p^m$ if $p = 2$, we have:*

(1) $\pi_{\mathrm{HT}}(\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_c) \subseteq B_r(1 : p\mathcal{O}_p)$,
(2) $\pi_{\mathrm{HT}}(\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a) \subseteq B_r(\mathcal{O}_p : 1)$.

For the proof, we need the following technical input on the Hodge–Tate morphism:

Proposition 5.19. *Let K be a completely valued extension of $\mathbb{Q}_p$ with algebraic closure $\overline{K}$. For any $v \in |\mathbb{R}|$, let $(p^v) := \{x \in K \mid |x| \leq v\}$. Let D be a p-divisible group over $\mathcal{O}_K$ of dimension d and height h. Let $0 \leq \epsilon$ be such that the Hodge ideal is $\mathrm{Hdg}(D) = (p^\epsilon)$. Let $n \in \mathbb{Z}_{\geq 0}$ be such that $\epsilon \leq 1/2p^{n-1}$ if $p \geq 5$, or $\epsilon \leq 1/3p^{n-1}$ if $p = 3$, or $\epsilon \leq 1/4p^{n-1}$ if $p = 2$. Let $\delta := \epsilon \frac{p^n - 1}{p-1} < 1$.*

(1) *D has canonical subgroups $1 \subseteq H_1 \subseteq \cdots \subseteq H_m \cdots \subseteq H_n \subseteq D$ of level m, finite locally free of rank p^{md}, for all $m \leq n$. They reduce to the kernel of Frobenius on D* mod $(p^{1-\delta})$.
(2) *The map $\omega_{D[p^n]} \to \omega_{H_n}$ induces an isomorphism $\omega_{D[p^n]}/(p^{n-\delta}) = \omega_{H_n}/(p^{n-\delta})$.*
(3) *The Hodge–Tate map $H_n(\overline{K})^\vee \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{K}} \to \omega_{H_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$ has cokernel of degree $\epsilon/(p-1)$.*

Proof. (1) is a special case of [AIP18, Cor. A.2, parts 1,2]. For (2), in the case of $p > 2$, this is [AIP15, Prop. 3.2.2]. For the case of $p = 2$, we can in the proof replace [AIP15, Thm. 3.1.1] by [AIP18, Cor. A.2.4]. Finally, for $p > 2$, (3) is again [AIP15, Prop. 3.2.2]. The case of (3) for $p = 2$ follows from [AIP18, Prop. A.3], which applies by $2 \in \mathrm{Hdg}(D)^4$ and which says that $\det \mathrm{coker} = \det \omega_{H_n}/\mathrm{HdgT}$ for an ideal $\mathrm{HdgT} \subseteq \mathcal{O}_K$ satisfying $\mathrm{HdgT}^{p-1} = \mathrm{Hdg}(D) = (p^\epsilon)$. □

Proof of Prop. 5.18. It suffices to check this on (C, C^+)-points for C algebraically closed.

Let $z \in \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)(C, C^+)$ correspond to a $\mathfrak{c}$-polarised HBAV A/C with extra data and an isomorphism $\alpha : \mathcal{O}_p^2 \to T_p A^\vee$. Let $A_0^\vee$ be the semi-abelian scheme over $\mathcal{O}_C$ associated to $A^\vee$ and let V be the kernel of the integral Hodge–Tate-map; then there is a left exact sequence

$$0 \to V \to T_p A_0^\vee \otimes \mathcal{O}_C \xrightarrow{\mathrm{HT}} \omega_{A_0}$$

and by definition, $V \subseteq T_p A_0^\vee \otimes \mathcal{O}_C$ is saturated. Via α, it thus gives a point $(a : b) \in \mathbb{P}^1(\mathcal{O}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_C) \cong \mathbb{P}^1(\mathcal{O}_C)^\Sigma$ with $a = (a_v)_v, b = (b_v)_v \in \mathcal{O}_C^\Sigma$, which is the image of z under π_{HT}.

Let $n := m + 1$. Upon reduction mod p^n, we get an injection $V/p^n \to A_0^\vee[p^n] \otimes_{\mathbb{Z}} \mathcal{O}_C$ which fits into a (not necessarily exact) complex

$$0 \to V/p^n \to A_0^\vee[p^n] \otimes \mathcal{O}_C \to \omega_{A_0[p^n]}.$$

The Hodge ideals of the p-divisible groups of A and $A^\vee$ are the same (e.g. [AIP15, Thm. 3.1.1]); thus by Prop. 5.19 (1) and our choice of ϵ, there is a canonical subgroup $H_n \subseteq A_0^\vee[p^n]$ of rank p^n. Modulo a certain power of p, the position of V/p^n coincides with that of H_n inside $A_0^\vee[p^n]$:

Claim. *Let $x = n - \frac{p^n}{p-1}\epsilon$. Then inside $A_0^\vee[p^n] \otimes \mathcal{O}_C/p^x$, we have $V/p^x = H_n \otimes \mathcal{O}_C/p^x$.*

To see that this proves the proposition, note that $H_n \otimes_{\mathbb{Z}} \mathcal{O}_C \subseteq A_0^\vee[p^n] \otimes_{\mathbb{Z}} \mathcal{O}_C$ has $\mathbb{Z}_p/p^n\mathbb{Z}_p$-coordinates. Moreover, the case (1) that $z \in \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_c(C, C^+)$ is equivalent to the coordinates of H_n being of the form $(1 : 0) \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}_p)^\Sigma$ after reducing modulo p. The claim then implies that $b/a \in p\mathcal{O}_p + p^x\mathcal{O}_C^\Sigma$, and hence $\pi_{\mathrm{HT}}(z) = (a : b) \in B_{|p^x|}(1 : p\mathcal{O}_p)$. Since $x > n - 1$, we have $|p^x| = 1/p^x \leq 1/p^{n-1} \leq r$. This implies $\pi_{\mathrm{HT}}(z) \in B_r(1 : p\mathcal{O}_p)$, as desired.

The proof of (2) follows in the same way, using that $z \in \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a(C, C^+)$ is equivalent to the coordinates of H_n being of the form $(c : 1) \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}_p)^\Sigma$ for some $c \in \mathbb{Z}/p\mathbb{Z}_p$, and therefore

$$\pi_{\mathrm{HT}}(z) = (a : b) \in B_{|p^x|}(\mathcal{O}_p : 1). \tag{5.4}$$

Proof of claim: Let $y := n - \delta = n - \frac{p^n-1}{p-1}\epsilon$. By Prop. 5.19 (2), modulo p^y the Hodge–Tate map can be described as

$$\mathrm{HT}_y : A_0^\vee[p^n] \otimes \mathcal{O}_C/p^y \to \omega_{A_0^\vee[p^n]} \otimes \mathcal{O}_C/p^y = \omega_{H_n} \otimes \mathcal{O}_C/p^y.$$

Let now $N := \ker \mathrm{HT}_y$ and $Q := \mathrm{coker}\, \mathrm{HT}_y$ and consider the exact sequence

$$0 \to N \to A_0^\vee[p^n] \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^y \xrightarrow{\mathrm{HT}_y} \omega_{H_n}/p^y \to Q \to 0$$

By Prop. 5.19 (3), the $\mathcal{O}_C$-module $Q = \mathrm{coker}\, \mathrm{HT}_y$ has degree $\partial := \epsilon/(p-1)$. Using additivity of degrees of $\mathcal{O}_C$-modules in extensions, we calculate that

$$\deg N = \deg(A_0[p^n] \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^y) - \deg \omega_{H_n}/p^y + \deg Q = 2gy - gy + \partial = gy + \partial.$$

Observe now that $M_1 := V/p^y$ and $M_2 := H_n \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^y$ are both free $\mathcal{O}_C/p^y$-submodules of rank g of $A_0[p^n] \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^y$ that are contained in N. Since N is p^y-torsion and of degree $gy + \partial$, we conclude from this that N is of the form $(\mathcal{O}_C/p^y)^g \oplus T$ as an $\mathcal{O}_C$-module, where T is p^∂-torsion. Second, this shows that inside $p^\partial N$, the modules $p^\partial M_1$ and $p^\partial M_2$ coincide. Thus the same is true inside $A_0^\vee[p^n] \otimes p^\partial \mathcal{O}_C/p^y$. Via multiplication by $p^\partial : \mathcal{O}_C/p^{y-\partial} \xrightarrow{\sim} p^\partial \mathcal{O}_C/p^y$, this shows that the images of M_1 and M_2 in $\mathcal{O}_C/p^{y-\partial}$ coincide. Since by definition $x = y - \partial$, this gives the desired statement, proving the claim, and hence the proposition. □

Definition 5.20. We write $\mathfrak{z}$ for the restriction of π_{HT} to the open subspaces

$$\mathfrak{z} : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to B_r(\mathcal{O}_p : 1) \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_a \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1.$$

Remark 5.21. If F is split in L, we also consider for any $v : F \to L$ the projection

$$\mathfrak{z}_v : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_a = \hat{\mathbb{G}}_a^\Sigma \xrightarrow{\pi_v} \hat{\mathbb{G}}_a.$$

By the universal property of $\hat{\mathbb{G}}_a$, we can interpret each $\mathfrak{z}_v$ as a function in $\mathcal{O}^+(\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a)$. However, we caution that for general L, the morphism $\mathfrak{z}$ admits no such canonical interpretation.

5.4. **The canonical differential.**

Definition 5.22. We define a G^*-equivariant vector bundle $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{O}(1)$ of rank g on $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1$ as follows: recall that on $\mathbb{P}^1$ we have the line bundle $\mathcal{O}(1)$ whose total space $\pi : \mathcal{T}(1) \to \mathbb{P}^1$ is therefore a $\mathbb{G}_m$-bundle with fibres $\mathbb{A}^1$. It moreover has a natural GL_2-equivariant action. By applying the functor $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}$, we see that $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \pi : \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1) \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1$ is a $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{G}_m$-bundle with fibres $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{A}^1$. As any choice of $\mathbb{Z}$-basis of $\mathcal{O}_F$ induces an isomorphism $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{A}^1 \cong \mathbb{A}^g$, we conclude that $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \pi$ is a vector bundle of rank g. It moreover receives a natural equivariant $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathrm{GL}_2 = G$-action (and hence a G^*-action) by functoriality.

Remark 5.23. The vector bundle $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1)$ has the following moduli interpretation: for any $\mathbb{Z}_p$-algebra R, the R-points of $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1$ parametrise quotients $R^2 \otimes_{\mathbb{Z}} \mathcal{O}_F \to Q$ of rank 1 as $R \otimes_{\mathbb{Z}} \mathcal{O}_F$-modules. Then $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1) \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1$ represents the choice of a point of Q.

Definition 5.24. Let $\omega_{\mathcal{A}}$ be the conormal sheaf of the universal abelian variety $\mathcal{A} \to \mathcal{X}$, an invertible $\mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{Z}} \mathcal{O}_F$-module. Its total space $\mathcal{T}(\omega_{\mathcal{A}}) \to \mathcal{X}$ is a $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{G}_m$-bundle. As before, if $q : \mathcal{X}_{K_p} \to \mathcal{X}$ is the forgetful map with K_p any of our wild levels, we let $\omega_{K_p} := q^*\omega_{\mathcal{A}}$.

As a special case of [CS17, Thm. 2.1.3. (2)], we then have the following result which forms the basis of our definition of Hilbert modular forms.

Proposition 5.25. *There is a* $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{G}_m$*-equivariant isomorphism*

$$\omega_{\Gamma^*(p^\infty)} = \pi_{\mathrm{HT}}^* \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{O}(1).$$

Recall that in §4 we have defined a canonical section $s : \mathbb{P}^1 \to \mathcal{T}(1)$ of $\mathcal{O}(1)$, non-vanishing away from ∞. We shall now change notation and denote this by $s_{\mathrm{ell}} : \mathbb{A}^1 \to \mathcal{T}(1)$. We now set:

Definition 5.26. Let $s := \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, s_{\mathrm{ell}} : \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1 \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1)$. This is a section of the vector bundle $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{O}(1)$, non-vanishing over $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{A}^1 \subseteq \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1$.

Remark 5.27. From the moduli description in the case of $g = 1$, we see that in the moduli interpretation, s sends a quotient $R^2 \otimes_{\mathbb{Z}} \mathcal{O}_F \to Q$ to the image of $(1, 0) \otimes 1$.

Remark 5.28. If F is split in L, we have $\mathcal{O}_F \otimes_{\mathbb{Z}} L = \prod_{v \in \Sigma} L$ where we interpret Σ as the set $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, L)$ and where the morphism into the v-component comes from the natural map $\mathcal{O}_F \otimes_{\mathbb{Z}} L \xrightarrow{v \otimes \mathrm{id}} L$. Consequently, we then get a canonical splitting $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1 = (\mathbb{P}^1)^\Sigma$. Similarly, we see on total spaces that the vector bundle $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{O}(1)$ becomes the direct sum $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{O}(1) = \bigoplus_{v \in \Sigma} \pi_v^* \mathcal{O}(1)$ of the pullbacks of $\mathcal{O}(1)$ on $\mathbb{P}^1$ along the projections $\pi_v : (\mathbb{P}^1)^\Sigma \to \mathbb{P}^1$. The section s then decomposes into partial sections $\mathfrak{s} = \sum_{v : F \hookrightarrow L} s_v$ where $s_v := \pi_v^* s_{\mathrm{ell}}$.

Lemma 5.29. *For any* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^*(p)$*, let* $(cz+d)$ *be the map* $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_a \xrightarrow{\cdot c} p\, \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_a \xrightarrow{+d} \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_m$*. Then we have* $\gamma^* s = (cz + d)s$*, in the sense that the following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_m \times \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1) & \xrightarrow{\mathrm{m}} & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1) & \xleftarrow{\gamma} & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1) \\
\uparrow{\scriptstyle (cz+d)\times s} & & \uparrow{\scriptstyle \gamma^* s} & & \uparrow{\scriptstyle s} \\
\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_a & \xrightarrow{\gamma} & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \hat{\mathbb{G}}_a & &
\end{array}$$

Proof. It suffices to show that this diagram commutes after extending L, so we may without loss of generality assume that F is split in L. Then by Rem. 5.28, $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathbb{P}^1 = (\mathbb{P}^1)^\Sigma$ is canonically split, as is the bundle $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\, \mathcal{T}(1) = \oplus_\Sigma \mathcal{O}(1)$, and the diagram becomes a product over Σ of the diagram in Lem. 3.19. □

Definition 5.30. Let $\mathfrak{s} := \pi_{\mathrm{HT}}^* s$. This is a section of $\pi_{\mathrm{HT}}^* \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathcal{O}(1) = \omega_{\Gamma^*(p^\infty)}$. Write $\mathcal{T}(\omega_{\Gamma^*(p^\infty)}) \to \mathcal{X}$ for the total space of $\omega_{\Gamma^*(p^\infty)}$; then we may regard $\mathfrak{s}$ as a morphism

$$\mathfrak{s} : \mathcal{X}_{\Gamma^*(p^\infty)} \to \mathcal{T}(\omega_{\Gamma^*(p^\infty)}).$$

As in the elliptic case, one checks that:

Lemma 5.31. *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^*(p)$, we write $c\mathfrak{z} + d$ for the composition*

$$c\mathfrak{z} + d : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \xrightarrow{\mathfrak{z}} \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a \xrightarrow{z \mapsto cz+d} \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_m.$$

Then we have $\gamma^* \mathfrak{s} = (c\mathfrak{z} + d)\mathfrak{s}$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a & \xrightarrow{\gamma} & \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \\ {\scriptstyle (c\mathfrak{z}+d)\times\mathfrak{s}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} \\ \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_m \times \mathcal{T}(\omega_{\mathcal{A}}) & \xrightarrow{\mathrm{m}} & \mathcal{T}(\omega_{\mathcal{A}}) \end{array}$$

Proof. This follows from Lem. 5.29 by pullback along π_{HT}. □

The crucial property of $\mathfrak{s}$ is given by the following moduli interpretation.

Lemma 5.32. *Let $x \in \mathcal{X}_{\Gamma^*(p^\infty)}(C, C^+)$ be a point corresponding to a HBAV A equipped with a $\Gamma^*(p^\infty)$-level $\alpha : \mathcal{O}_p^2 \xrightarrow{\sim} T_p A^\vee$ and extra structures. Then via $\pi_{\mathrm{HT}}^* \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathcal{O}(1) = \omega_{\Gamma^*(p^\infty)}$,*

$$\mathfrak{s}(x) = \mathrm{HT}_A(\alpha(1,0)) \in \omega_A.$$

Proof. In terms of the total spaces $\mathcal{T}(\omega_{\Gamma^*(p^\infty)}) \to \mathcal{X}_{\Gamma^*(p^\infty)}$ and $\operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathcal{T}(1) \to \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{P}^1$, by Rem. 5.15 the isomorphism $\omega_{\Gamma^*(p^\infty)} = \pi_{\mathrm{HT}}^* \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathcal{O}(1)$ is defined in the fibre of x by sending

$$(A, \alpha, \eta \in \omega_A) \mapsto (\mathcal{O}_p^2 \otimes_{\mathbb{Z}_p} C \xrightarrow{\alpha} T_p A^\vee \otimes_{\mathbb{Z}_p} C \xrightarrow{\mathrm{HT}} \omega_A, \eta \in \omega_A).$$

Since s by Rem. 5.27 sends a quotient $x : \mathcal{O}_p^2 \otimes_{\mathbb{Z}_p} C \to Q$ to the image of $(1,0) \otimes 1$ under x, it follows that $\mathfrak{s}$ sends x to the image of $(1,0)$ under $\mathrm{HT} \circ \alpha$. □

6. Geometric overconvergent Hilbert modular forms

6.1. **Weights and analytic continuation.** Next we define the relevant weight spaces for G^* and G, and set up some notational conventions as to how they are related.

Definition 6.1. Let $\mathbb{T} := \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \mathbb{G}_m$, then define:

(i) $\mathcal{W} := \operatorname{Spf}(\mathbb{Z}_p[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!])_\eta^{\mathrm{an}} \times L$, the *weight space for G*.
(ii) $\mathcal{W}^* := \operatorname{Spf}(\mathbb{Z}_p[\![\mathbb{T}(\mathbb{Z}_p)]\!])_\eta^{\mathrm{an}} \times L$, the *weight space for G^**.

An L-point $(w,t) \in \mathcal{W}(L)$ is a pair of maps $w : \mathbb{T}(\mathbb{Z}_p) \to L^\times$ and $t : \mathbb{Z}_p^\times \to L^\times$ (and analogously, an L-point of $\mathcal{W}^*$ is just a map $\mathbb{T}(\mathbb{Z}_p) \to L^\times$). Following [AIP16b], we let $\rho : \mathcal{W} \to \mathcal{W}^*$ be the morphism associated to the map $\mathbb{T}(\mathbb{Z}_p) \to \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ defined by $x \mapsto (x^2, N_{F/\mathbb{Q}}(x))$. For $(w,t) \in \mathcal{W}(\mathbb{C}_p)$ we write $\kappa = w^2 \cdot (t^{-1} \circ N_{F/\mathbb{Q}})$ for its image in $\mathcal{W}^*(\mathbb{C}_p)$, noting that $\kappa(x) \cdot w(x^{-2})$ factors through some power of the norm.

Definition 6.2. In order to be able to treat single weights and families in a uniform way, we define a weight to be a morphism $\kappa : \mathcal{U} \to \mathcal{W}$ or $\kappa : \mathcal{U} \to \mathcal{W}^*$ for G and G^* respectively, where $\mathcal{U}$ is a smooth rigid space over some perfectoid field extension of L. We say that κ is bounded if its image in $\mathcal{W}$ or $\mathcal{W}^*$ is contained in some affinoid open subspace. This generalises Defn. 3.1.

By unravelling the definitions, a weight $\kappa : \mathcal{U} \to \mathcal{W}^*$ determines a morphism

$$\kappa : \mathcal{O}_p^\times \times \mathcal{U} \to \hat{\mathbb{G}}_m,$$

which in an abuse of notation we also denote κ. The weight κ is then bounded if and only if

$$|T_\kappa| := \sup_{(t,x) \in \mathcal{O}_p^\times \times \mathcal{U}} |\kappa(t,x) - 1| < 1$$

Similarly, for G we have associated to any $\kappa : \mathcal{U} \to \mathcal{W}$ a pair of maps (w_κ, t_κ) of the form $w_\kappa : \mathcal{O}_p^\times \times \mathcal{U} \to \hat{\mathbb{G}}_m$ and $t_\kappa : \mathbb{Z}_p^\times \times \mathcal{U} \to \hat{\mathbb{G}}_m$. By composing with ρ, we get an associated weight $\rho(w_\kappa, t_\kappa) = w_\kappa^2 \cdot (t_\kappa^{-1} \circ N_{F/\mathbb{Q}})$ for G^*, which we use to see any weight for G as a weight for G^*.

Recall from Definition 5.17 that we embed $\mathcal{O}_p$ as a profinite set into $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\mathbb{P}^1$ by sending $z \mapsto (z:1)$. Given a bounded weight, one can then always find an analytic continuation of κ to a neighbourhood of $\mathcal{O}_p^\times$ in $\mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\mathbb{P}^1$. More precisely:

Proposition 6.3. *Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a bounded smooth weight. Let $r_0 = 1$ if $p > 2$ and $r_0 = 3$ if $p = 2$. Let $r_\kappa := |p|^{r_0}|T_\kappa|$, then for $r_\kappa \geq r > 0$, the morphism κ extends uniquely to a morphism*

$$\kappa^{\mathrm{an}} : B_r(\mathcal{O}_p^\times : 1) \times \mathcal{U} \to \hat{\mathbb{G}}_m.$$

Proof. We first prove that such a bound exists. In case that F is split in L, this is completely analogous to Prop. 3.12. In general, we first pass to a finite Galois extension $L'|L$, with group H and in which F is split, to obtain a morphism $B_r(\mathcal{O}_p^\times : 1) \times_L L' \times \mathcal{U} \to \hat{\mathbb{G}}_m$. Passing to the quotient by H, the result follows.

The precise value of r_κ follows from [AIP16a, Prop. 2.8]. □

Definition 6.4. Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a smooth bounded weight. Let $\epsilon_\kappa^{\mathrm{def}} > 0$ be such that $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon_\kappa^{\mathrm{def}})_a \subseteq B_{r_\kappa}(\mathcal{O}_p : 1)$, see Prop. 5.18 for a precise bound on $\epsilon_\kappa^{\mathrm{def}}$. Then for any $c \in p\mathcal{O}_p$, $d \in \mathcal{O}_p^\times$, we define the invertible function $\kappa(c\mathfrak{z} + d) \in \mathcal{O}^+(\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a)^\times$ to be the composition

$$\kappa(c\mathfrak{z}+d) : \mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a \xrightarrow{\pi_{\mathrm{HT}}\times\mathrm{id}} B_r(\mathcal{O}_p:1)\times\mathcal{U} \xrightarrow{(cz+d)\times\mathrm{id}} B_r(\mathcal{O}_p^\times:1)\times\mathcal{U} \xrightarrow{\kappa^{\mathrm{an}}} \hat{\mathbb{G}}_m,$$

where $\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a := \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \times_L \mathcal{U}$.

6.2. Definition of overconvergent Hilbert modular forms.

Definition 6.5. For $\kappa : \mathcal{U} \to \mathcal{W}^*$ a bounded smooth weight, $0 \leq \epsilon \leq \epsilon_\kappa^{\mathrm{def}}$ and $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, we define a sheaf ω_n^κ on $\mathcal{X}_{\mathcal{U},\Gamma_0^*(p^n)}(\epsilon)_a$ by setting

$$\omega_n^\kappa(U) := \{f \in q_*\mathcal{O}_{\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a}(U) | \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0^*(p^n)\},$$

where $q : \mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathcal{U},\Gamma_0^*(p^n)}(\epsilon)_a$ is the projection. We similarly get the integral subsheaf

$$\omega_n^{\kappa,+}(U) := \{f \in q_*\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a}(U) | \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0^*(p^n)\},$$

by using the $\mathcal{O}^+$-sheaf instead. For $n = 0$, as before, via the Atkin–Lehner isomorphism $\mathrm{AL} : \mathcal{X}_{\mathcal{U},\Gamma_0^*(p)}(p\epsilon)_a \xrightarrow{\sim} \mathcal{X}_{\mathcal{U}}(\epsilon)$ we define the sheaves $\omega^\kappa := \omega_0^\kappa := \mathrm{AL}_*\omega_1^\kappa$ and $\omega^{\kappa,+} := \omega_0^\kappa := \mathrm{AL}_*\omega_1^{\kappa,+}$ on $\mathcal{X}_{\mathcal{U}}(\epsilon)$ thus giving a sheaf on the tame level Hilbert modular variety. If needed we will add a subscript G^* to make clear these are sheaves for G^*.

Exactly like in Prop. 3.24, we see:

Proposition 6.6. *ω_n^κ is an analytic line bundle on $\mathcal{X}_{\mathcal{U}}(\epsilon)$.*

We will also see this in Thm. 7.14, which moreover shows that $\omega_n^{\kappa,+}$ is an invertible $\mathcal{O}^+$-modules.

Proof. Exactly as in the elliptic case, [Heub, Cor. 4.1] shows that the analyticity overconverges if we can prove it for $\epsilon = 0$. By the same argument, we may restrict to the good reduction locus, as this is Zariski-dense in $\mathcal{X}_{\mathcal{U}}$. Over this, we again have an Igusa tower with a pro-étale formal model, and like in the elliptic case, [Heub, Prop. 4.8] gives the desired statement. □

Warning 6.7. We caution the reader that ω_n^1 is not the same as $\omega_{\Gamma_0^*(p^n)}$ from Defn. 5.24, as the latter is not an invertible sheaf, when $[F : \mathbb{Q}] > 1$. Instead, we have $\omega_n^1 = \det \omega_{\Gamma^*(p^n)}$.

Definition 6.8. Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a bounded smooth weight, $0 \leq \epsilon \leq \epsilon_\kappa^{\mathrm{def}}$ and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We define the space of $\mathfrak{c}$-polarised overconvergent Hilbert modular forms for G^* of weight κ, wild level $\Gamma_0^*(p^n)$, tame level μ_N and radius of overconvergence ϵ to be the L-vector space

$$M_\kappa^{G^*}(\Gamma_0^*(p^n), \mu_N, \epsilon, \mathfrak{c}) := \mathrm{H}^0(\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma_0^*(p^n),\mu_N}(\epsilon)_a,\ \omega_{G^*,n}^\kappa).$$

Similarly, we define the space of integral overconvergent Hilbert modular forms for G^* to be

$$M_\kappa^{G^*,+}(\Gamma_0^*(p^n), \mu_N, \epsilon, \mathfrak{c}) := \mathrm{H}^0(\mathcal{X}_{\mathfrak{c},\mathcal{U},\Gamma_0^*(p^n),\mu_N}(\epsilon)_a,\ \omega_{G^*,n}^{\kappa,+}).$$

Remark 6.9. By the Koecher principle (see [AIP16a, Prop. 8.4]) or [AIP16b, Theorem 5.5.1], $\omega_{\mathfrak{c}}^\kappa$ extends uniquely to a line bundle on a suitable toroidal compactification $\mathcal{X}^{\mathrm{tor}}(\mathfrak{c})$. Let ∂

denote the boundary divisor, then one can define the subspaces of cusp forms as sections of the subsheaf $\omega^\kappa_\mathfrak{c}(-\partial)$. Via Thm. 7.14 below, these agrees with the spaces of cusp forms defined in [AIP16b]. In particular, they will be projective Banach modules with surjective specialisation maps (see [AIP16a, Thm. 3.16]).

7. Comparison to Andreatta–Iovita–Pilloni's geometric Hilbert modular forms

In this section, we will show that our spaces of overconvergent Hilbert modular forms for G^* coincide with those defined in [AIP16a].

7.1. **The Andreatta–Iovita–Pilloni-torsor.** Like in the elliptic case, Andreatta–Iovita–Pilloni construct integral sheaves of Hilbert modular forms on the Hilbert modular variety as a formal scheme over $\mathcal{O}_F$. In order to define such a sheaf on the full Hilbert modular variety over $\mathcal{O}_F$, the definition of the Pilloni-torsor in the Hilbert case is not just the straightforward adaptation of the elliptic case (the issue appears away from the Rapoport locus, i.e. on the closed subscheme concentrated in the special fibre where the abelian scheme does not satisfy the Rapoport condition). Instead, Andreatta–Iovita–Pilloni in [AIP16a, §4.1] explain how this definition needs to be modified by endowing the sheaf $\omega_\mathcal{A}$ with an integral structure ω^{int} (denoted by $\mathcal{F}$ *op. cit.*) which, when p is ramified in F, is different to the canonical one. We briefly recall the construction, with the minor modification that as before we present it in the analytic setting over L rather than in the excellent Noetherian setting of [AIP16a, §4.1].

Definition 7.1. (1) For any $m \in \mathbb{Z}_{\geq 1}$, let $\epsilon_m^{\mathrm{can}} := 1/p^{m+1}$ as before. Then [AIP18, Cor. A.2] implies that, for $0 \leq \epsilon \leq \epsilon_m^{\mathrm{can}}$, the universal semi-abelian variety $\mathcal{A}$ on $\mathcal{X}(\epsilon)$ admits a canonical subgroup $H_m \subseteq \mathcal{A}$ of order p^m, étale locally isomorphic to $\mathcal{O}_F/p^m\mathcal{O}_F$.

(2) We denote by $\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon) \to \mathcal{X}(\epsilon)$ the finite étale $(\mathcal{O}_F/p^m\mathcal{O}_F)^\times$-torsor which relatively represents isomorphisms $\mathcal{O}_F/p^m\mathcal{O}_F \to H_m^\vee$ of adic spaces with $\mathcal{O}_F$-module structure

Let $\omega_{\mathrm{Ig}(p^m)}$ be the conormal sheaf of the pullback of $\mathcal{A}$ to $\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)$. It has an integral subsheaf $\omega^+_{\mathrm{Ig}(p^m)}$ obtained from its formal model on $\mathfrak{X}^*(\epsilon)$. The canonical subgroup $H_m \subseteq \mathcal{A}$, considered as a finite flat group over $\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)$ induces a map $\pi : \omega^+_{\mathrm{Ig}(p^m)} \to \omega^+_{H_m}$. As in Lem. 4.3, we see:

Lemma 7.2 ([AIP18, Cor. A.4])**.** *We have a right exact sequence of $\mathcal{O}^+_{\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)}$-modules*

$$I_m \cdot \omega^+_{\mathrm{Ig}(p^m)} \to \omega^+_{\mathrm{Ig}(p^m)} \xrightarrow{\pi} \omega^+_{H_m} \to 0, \quad \text{where} \quad I_m := p^m \,\mathrm{Hdg}^{-\frac{p^m-1}{p-1}}.$$

The Hodge–Tate map now defines a morphism of sheaves of $\mathcal{O}_F$-modules over $\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)$

$$\psi : \mathcal{O}_F/p^m\mathcal{O}_F \to H_m^\vee \xrightarrow{\mathrm{HT}} \omega^+_{H_m} \to \omega^+_{\mathrm{Ig}(p^m)}/I_m.$$

Definition 7.3. Let $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}$ be the $\mathcal{O}_F \otimes_\mathbb{Z} \mathcal{O}^+_{\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)}$-submodule of $\omega^+_{\mathrm{Ig}(p^m)}$ defined as the preimage of the $\mathcal{O}_F$-submodule of $\omega^+_{\mathrm{Ig}(p^m)}/I_m$ generated by $\psi(1)$.

The sheaf $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}$ gives a second integral structure on $\omega_{\mathrm{Ig}(p^m)}$. If p is ramified in $\mathcal{O}_F$, it is better behaved than $\omega^+_{\mathrm{Ig}(p^m)}$, because it always satisfies the analogue of the Rapoport condition:

Proposition 7.4 ([AIP16a, Prop. 4.1])**.**

(1) The sheaf $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}$ is a locally free $\mathcal{O}_F \otimes_\mathbb{Z} \mathcal{O}^+_{\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)}$-module on $\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)$.

(2) The cokernel of $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)} \subseteq \omega^+_{\mathrm{Ig}(p^m)}$ is annihilated by $\mathrm{Hdg}^{1/(p-1)}$. We thus have an injection

$$\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}/I_m \hookrightarrow \omega^+_{\mathrm{Ig}(p^m)}/I_m$$

whose image is precisely the $\mathcal{O}_F \otimes_\mathbb{Z} \mathcal{O}^+_{\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)}$-submodule generated by $\psi(1)$.

(3) Let $I'_m := p^m \,\mathrm{Hdg}^{-\frac{p^m}{p-1}} \supseteq I_m$. Sending $1 \mapsto \psi(1)$ induces an isomorphism

$$\mathrm{HT}' : \mathcal{O}_F \otimes_\mathbb{Z} \mathcal{O}^+_{\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)}/I'_m \xrightarrow{\sim} \omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}/I'_m.$$

Definition 7.5. The Andreatta–Iovita–Pilloni-torsor is the subsheaf of $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}$ defined by

$$\mathfrak{F}_m := \{w \in \omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)} | w \equiv \mathrm{HT}'(1) \bmod I'_m\}.$$

We denote the analytic total space of $\mathfrak{F}_m \subseteq \omega^{+}_{\mathrm{Ig}(p^n)}$ over $\mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)$ by

$$\mathcal{F}_m(\epsilon) \to \mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon).$$

By Prop. 7.4.(3), this is a torsor in the analytic topology for the subgroup $1 + I'_m \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a|_{\mathcal{X}(\epsilon)}$ of $\operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_m|_{\mathcal{X}(\epsilon)}$, where $\operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a|_{\mathcal{X}(\epsilon)}$ is the pullback of $\operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a \to \mathrm{Spa}(L)$ to $\mathcal{X}(\epsilon)$, and similarly for $\operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_m|_{\mathcal{X}(\epsilon)}$. In particular, the composition $\mathcal{F}_m(\epsilon) \to \mathcal{X}(\epsilon)$ is a torsor in the étale topology for the subgroup $B_m := \mathcal{O}_p^\times(1 + I'_m \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a|_{\mathcal{X}(\epsilon)}) \subseteq \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_m|_{\mathcal{X}(\epsilon)}$. This also shows that the natural map $\mathcal{F}_m(\epsilon) \to \mathcal{T}(\omega)$ into the total space of ω over $\mathcal{X}$ is an open immersion. Finally, we note that since $p^\epsilon \in \mathrm{Hdg}$, we have for $x := m - \epsilon p^m/(p-1)$ that

$$\mathcal{O}_p^\times(1 + p^x \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a)|_{\mathcal{X}(\epsilon)} \subseteq B_m. \tag{7.1}$$

The following corollary relates $\mathfrak{F}_m$ to the definition in the elliptic case:

Corollary 7.6. *Let $w \in \omega^{+}_{\mathrm{Ig}(p^m)}$ be any lift of $\psi(1) \in \omega^{+}_{H_m}$ under $\omega^{+}_{\mathrm{Ig}(p^m)} \to \omega^{+}_{H_m}$. Then $w \in \mathfrak{F}_m$.*

Proof. By Lem. 7.2, for w to be a lift of $\psi(1)$ means that w and the image of $\psi(1)$ in $\omega^{+}_{\mathrm{Ig}(p^m)}/I_m$ agree. Thus $w \in \omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}$. By the injective morphism from Proposition 7.4.(2), this shows that $\psi(1)$ and w also agree in $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}/I_m$ and thus in its quotient $\omega^{\mathrm{int}}_{\mathrm{Ig}(p^m)}/I'_m$. This means $w \in \mathfrak{F}_m$. □

Definition 7.7. Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a bounded smooth weight. Recall that we may regard κ as a morphism $\mathcal{O}_p^\times \times U \to \hat{\mathbb{G}}_m$. As before, we let $|\delta_\kappa| := \max(|p|, |T_\kappa|)$. Let $r = 3$ if $p > 2$ and $r = 5$ if $p = 2$. Let $\epsilon_\kappa > 0$ be implicitly defined by $|p|^{\epsilon_\kappa} = |\delta_\kappa|^{1/p^{r+1}}$. We note that $\epsilon_\kappa \leq \epsilon_\kappa^{\mathrm{def}}$.

Definition 7.8. For any $k \in \mathbb{Z}_{\geq 1}$, let $\mathcal{W}_k^*$ be the open in weight space denoted by $\mathcal{W}_{F,[p^{k-1},p^k]}$ in [AIP16a, §2], and let $\mathcal{W}_0^*$ be the open denoted by $\mathcal{W}_{F,[0,1]}$. Explicitly, for any $k \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{W}_k^* := \mathcal{W}^*(|\delta_\kappa|^{p^k} \leq |p| \leq |\delta_\kappa|^{p^{k-1}})$. Then $\mathcal{W}^* = \cup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{W}_k^*$.

If $\kappa : \mathcal{U} \to \mathcal{W}^*$ is a bounded smooth weight, we let $\mathcal{U}_k := \kappa^{-1}(\mathcal{W}_k^*)$, then $\mathcal{U} = \cup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{U}_k$.

Definition 7.9. Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a smooth bounded weight and let $0 \leq \epsilon \leq \epsilon_\kappa$. For each $k \in \mathbb{Z}_{\geq 0}$, let $m = k + r$ (this is the variable "n" in [AIP16a]), so that $\epsilon_\kappa \leq \epsilon_m^{\mathrm{can}}$. The sheaf $\omega^\kappa_{\mathrm{AIP}}$ on $\mathcal{X}(\epsilon) \times \mathcal{U}_k$ of modular forms of weight κ, as defined in [AIP16a], is given locally as

$$\omega^\kappa_{\mathrm{AIP}|\mathcal{U}_k} := \mathcal{O}_{\mathcal{F}_m(\epsilon) \times \mathcal{U}_k}[\kappa^{-1}] = \left\{ f \in \mathcal{O}_{\mathcal{F}_m(\epsilon) \times \mathcal{U}_k} | \gamma^* f = \kappa^{-1}(\gamma) f \text{ for all } \gamma \in B_m \right\}.$$

As usual, we define an integral subspace $\omega^{\kappa,+}_{\mathrm{AIP}|\mathcal{U}_k}$ by using $\mathcal{O}^+$ instead.

Proposition 7.10 ([AIP16a, Prop. 4.3])**.** *Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a smooth bounded weight and let $0 \leq \epsilon \leq \epsilon_\kappa$. Then the sheaves $\omega^{\kappa,+}_{\mathrm{AIP}|\mathcal{U}_k}$ can be canonically identified on intersections $\mathcal{X}(\epsilon) \times (\mathcal{U}_k \cap \mathcal{U}_{k+1})$, so that they glue to give an $\mathcal{O}^+$-module $\omega^{\kappa,+}_{\mathrm{AIP}}$ on $\mathcal{X}_{\mathcal{U}}(\epsilon) = \mathcal{X}(\epsilon) \times \mathcal{U}$. This $\mathcal{O}^+$-module is invertible. Similarly, we can glue the $\omega^\kappa_{\mathrm{AIP}|\mathcal{U}_k}$ to get a line bundle $\omega^\kappa_{\mathrm{AIP}}$ on $\mathcal{X}_{\mathcal{U}}(\epsilon)$.*

Proof. We first need to explain how our variable ϵ is related to the radius variable r used in [AIP16a, §3]. The sheaf "$\mathfrak{w}_{n,r,I}$" constructed in [AIP16a, §4] for $I = [p^{k-1}, p^k]$ lives on a formal scheme "$\mathfrak{X}_{r,[p^{k-1},p^k]}$" over $\mathbb{Z}_p$. The base-change to L of the generic fibre of this formal scheme is the subspace of $\mathcal{X} \times \mathcal{W}_k^*$ cut out by the condition $|\mathrm{Ha}^{p^{r+1}}| \geq |\delta_\kappa|$. Since we have set $|p|^{\epsilon_\kappa} = |\delta_\kappa|^{1/p^{r+1}}$, this is contained in $\mathcal{X}(\epsilon) \times \mathcal{W}_k^*$. By [AIP16a, Prop. 4.3], the sheaf "$\mathfrak{w}_{n,r,I}$" is a line bundle. It now follows from our definition that the sheaf $\omega^{\kappa,+}_{\mathrm{AIP}|\mathcal{U}_k}$ is the pullback of "$\mathfrak{w}_{n,r,I}$" along the morphism of ringed spaces $(\mathcal{X}(\epsilon) \times \mathcal{U}_k, \mathcal{O}^+) \to \mathfrak{X}_{r,k}$. In particular, $\omega^{\kappa,+}_{\mathrm{AIP}|\mathcal{U}_k}$ is invertible.

By [AIP16a, Prop. 4.7], there is a canonical isomorphism between this sheaf and the one defined using $m+1$ instead of m. This shows that the $\omega^{\kappa,+}_{\mathrm{AIP}|\mathcal{U}_k}$ glue on intersections $\mathcal{U}_k \cap \mathcal{U}_{k+1}$. Since $\omega^{\kappa,+}_{\mathrm{AIP}|\mathcal{U}_k}$ is invertible on each open $\mathcal{X}(\epsilon) \times \mathcal{U}_k$, it is clearly also invertible on $\mathcal{X}_{\mathcal{U}}(\epsilon)$. □

7.2. **The comparison morphism.** Recall from Defn. 5.30 that over $\mathcal{X}_{\Gamma(p^\infty)}$, the conormal sheaf $\omega_{\Gamma^*(p^\infty)}$ has a canonical section $\mathfrak{s}$ that we may regard as a morphism into the total space $\mathfrak{s} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{T}(\omega_{\Gamma^*(p^\infty)}) \to \mathcal{T}(\omega)$. Let us simply write $\mathcal{T}(\epsilon) \to \mathcal{X}(\epsilon)$ for the restriction of $\mathcal{T}(\omega)$ to $\mathcal{X}(\epsilon)$, then $\mathfrak{s}$ restricts to

$$\mathfrak{s} : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{T}(\epsilon).$$

The comparison of our Hilbert modular forms to the ones defined by [AIP16a] relies on the following proposition saying that $\mathfrak{s}$ compares $\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a$ to the Andreatta–Iovita–Pilloni-torsor.

Proposition 7.11. *Let $0 \leq \epsilon \leq \epsilon_m^{\mathrm{can}}$. Then $\mathfrak{s} : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{T}(\epsilon)$ factors through the subspace*

$$\mathfrak{s} : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{F}_m(\epsilon) \subset \mathcal{T}(\epsilon)$$

defined by the Andreatta–Iovita–Pilloni torsor (see Definition 7.5).

Proof. It suffices to check that for C any complete algebraically closed extension of L, the (C, C^+)-points that the image of $\mathfrak{s}$ are contained in the open subspace $\mathcal{F}_m(\epsilon) \subseteq \mathcal{T}(\epsilon)$.

There is a natural map $\varphi : \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathrm{Ig}(p^m)}(\epsilon)$ defined by sending a point valued in some stably uniform adic ring (R, R^+) corresponding to an abelian variety A and a trivialisation $\alpha : \mathcal{O}_p^2 \xrightarrow{\sim} T_p A^\vee$ to the trivialisation of $H^\vee_m(A)$ given by the composition

$$\mathcal{O}_p/p^m\mathcal{O}_p \xrightarrow{(1,0)} (\mathcal{O}_p/p^m\mathcal{O}_p)^2 \xrightarrow{\alpha \bmod p^m} A^\vee[p^m] \to H^\vee_m$$

with the dual of the inclusion $H_n \to A[p^m]$, where as usual we identify $A[p^m]^\vee = A^\vee[p^m]$ via the Weil pairing. By functoriality of the Hodge–Tate map we then have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_p^2 & \xrightarrow{\alpha} & T_pA^\vee & \xrightarrow{\mathrm{HT}} & \omega_A^+ \\ {\scriptstyle (1,0) \bmod p^n}\downarrow & & \downarrow & & \downarrow \\ \psi : \mathcal{O}_p/p^m\mathcal{O}_p & \longrightarrow & H^\vee_m & \xrightarrow{\mathrm{HT}} & \omega^+_{H_m}. \end{array}$$

For $(R, R^+) = (C, C^+)$, we then have $\mathfrak{s}(x) = \mathrm{HT} \circ \alpha(1, 0)$ by Lem. 5.32. This shows that $\mathfrak{s}(x)$ is a lift of $\psi(1) \in \omega^+_{H_m}$. By Cor. 7.6, this implies that $\mathfrak{s}(x) \in \mathcal{F}_m(\epsilon)(C, C^+)$ as desired. □

Lemma 7.12. *The following diagram commutes*

$$\begin{array}{ccccc} \Gamma_0^*(p) \times \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a & \xrightarrow{\mathrm{m}} & \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}_{\Gamma_0^*(p^\infty)}(\epsilon)_a \\ {\scriptstyle (c\mathfrak{z}+d)\times\mathfrak{s}}\downarrow & & {\scriptstyle \mathfrak{s}}\downarrow & & \downarrow{\scriptstyle q} \\ B_m \times \mathcal{F}_m(\epsilon) & \xrightarrow{\mathrm{m}} & \mathcal{F}_m(\epsilon) & \longrightarrow & \mathcal{X}(\epsilon). \end{array}$$

Proof. We first note that the morphism $\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to B_m$ is well-defined: for this we use that by (7.1), we have $\mathcal{O}_p^\times(1 + p^x \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a) \mid_{\mathcal{X}(\epsilon)} \subseteq B_m$. Moreover, by (5.4), the map $\mathfrak{z}$ already restricts to $\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to B_{|p^x|}(\mathcal{O}_p : 1) = \mathcal{O}_p^\times(1 + p^x \operatorname{Res}_{\mathcal{O}_F|\mathbb{Z}} \hat{\mathbb{G}}_a)$. The left square now commutes by Lem. 5.31. Commutativity of the right square is clear. □

Combining this with the morphism $u_n : \mathcal{X}_{\Gamma^*(p^\infty)}(p^n\epsilon)_a \to \mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a$ defined by the action of the matrix $\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we obtain from the lemma a commutative diagram

$$\begin{array}{ccccc} \Gamma_0^*(p) \times \mathcal{X}_{\Gamma^*(p^\infty)}(p^n\epsilon)_a & \xrightarrow{\mathrm{m}} & \mathcal{X}_{\Gamma^*(p^\infty)}(p^n\epsilon)_a & \longrightarrow & \mathcal{X}_{\Gamma_0^*(p^n)}(p^n\epsilon)_a \\ {\scriptstyle (c\mathfrak{z}+d)\times\tilde{\mathfrak{s}}}\downarrow & & {\scriptstyle \tilde{\mathfrak{s}}}\downarrow & & \wr\downarrow{\scriptstyle \mathrm{AL}^n} \\ B_m \times \mathcal{F}_m(\epsilon) & \xrightarrow{\mathrm{m}} & \mathcal{F}_m(\epsilon) & \longrightarrow & \mathcal{X}(\epsilon) \end{array} \tag{7.2}$$

where $\tilde{\mathfrak{s}} = \mathfrak{s} \circ u_n$. From this, we finally deduce the following Hilbert analogue of Thm. 4.8.

Definition 7.13. (1) For any $n \in \mathbb{Z}_{\geq 0}$ we set $\omega^{\kappa,+}_{\mathrm{AIP},n} := \mathrm{AL}^{n*}\omega^{\kappa,+}_{\mathrm{AIP}}$ where

$$\mathrm{AL}^n : \mathcal{X}_{\Gamma_0^*(p^n)}(p^n\epsilon)_a \xrightarrow{\sim} \mathcal{X}(\epsilon)$$

is the Atkin–Lehner isomorphism. By [AIP16a, Thm. 6.7.3], the restriction of $\omega^{\kappa,+}_{\mathrm{AIP},n}$ to $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ equals $q_n^*\omega^{\kappa,+}_{\mathrm{AIP}}$, where $q_n : \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a \to \mathcal{X}(\epsilon)$ is the forgetful map.

(2) We then set $\omega^{\kappa,+}_{\mathrm{AIP},\infty} := q^*\omega^{\kappa,+}_{\mathrm{AIP}}$ where $q : \mathcal{X}_{\Gamma_0^*(p^\infty)}(\epsilon)_a \to \mathcal{X}(\epsilon)$ is the forgetful map.

Theorem 7.14. *Let $\kappa : \mathcal{U} \to \mathcal{W}^*$ be a bounded smooth weight. Let $0 \leq \epsilon \leq \epsilon_\kappa$. Then for any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the map $\tilde{\mathfrak{s}}^*$ induces a Hecke-equivariant isomorphism of $\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma_0(p^n)(\epsilon)_a}}$-modules*

$$\tilde{\mathfrak{s}}^* : \omega_n^{\kappa,+} \xrightarrow{\sim} \omega^{\kappa,+}_{\mathrm{AIP},n}.$$

In particular, $\omega_n^{\kappa,+}$ is an invertible $\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma_0(p^n)(\epsilon)_a}}$-module, and $\omega^\kappa \cong \omega^\kappa_{\mathrm{AIP}}$.

Proof. With the preparations from this and the last section, we can argue as in the elliptic case. It suffices to prove this locally on $\mathcal{W}^*$. We may therefore without loss of generality assume that κ has image in $\mathcal{W}^*_k$ for some $k \in \mathbb{Z}_{\geq 1}$ and set $m := k + r$.

We first check that $\mathfrak{s}$ induces a map $\omega^{\kappa,+}_{\mathrm{AIP},n} \to \omega_n^{\kappa,+}$. This is because for any section f of $\omega^{\kappa,+}_{\mathrm{AIP},n}$ and for any $\gamma \in \Gamma_0^*(p^n)$, diagram (7.2) implies that the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(p^n\epsilon)_a & \xrightarrow{(c\mathfrak{z}+d)\times\tilde{\mathfrak{s}}} & B_m \times \mathcal{F}_m(\epsilon) & \xrightarrow{\kappa^{-1}\times f} & \mathbb{G}_m \times \mathbb{A}^{\mathrm{an}} \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{m}} & & \downarrow{\scriptstyle \mathrm{m}} \\
\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(p^n\epsilon)_a & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{F}_m(\epsilon) & \xrightarrow{f} & \mathbb{A}^{\mathrm{an}},
\end{array}$$

As before, this together with $\omega^{\kappa,+}_{\mathrm{AIP},n}$ being invertible proves the theorem for $n \in \mathbb{Z}_{\geq 1}$ since

$$(\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)(\epsilon)_a}})^{\Gamma_0^*(p^n)} = \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma_0^*(p^n)(\epsilon)_a}}$$

by Lem. 3.7. The case of $n = \infty$ follows by the same argument from the diagram in Lem. 7.12. The case of $n = 0$ follows from $\omega^{\kappa,+} = \mathrm{AL}^*\omega_1^{\kappa,+} = \omega^{\kappa,+}_{\mathrm{AIP},1}$. Finally, the isomorphism $\omega^\kappa = \omega^\kappa_{\mathrm{AIP}}$ is induced from the integral one by inverting p.

We postpone the proof of Hecke equivariance to §10 where we discuss the Hecke action. □

8. Perfectoid Hilbert modular varieties for G

We now pass from modular forms for G^* to those for G, the so-called arithmetic Hilbert modular forms. This requires a closer study of the perfectoid modular varieties attached to G, and their relation to the perfectoid modular varieties of G^*, which is the subject of this section.

Notation 8.1. Recall that in §5.1.2 we have defined Hilbert modular varieties X, X_G over L which are base-changes of models for the Shimura varieties attached to G^* and G respectively. In doing so, we had fixed a choice of tame level μ_N as well as polarisation ideal $\mathfrak{c}$ and omit these notation. We denote by $\mathcal{X}, \mathcal{X}_G$ the adic analytifications of X, X_G.

We begin by recalling that the action of $\mathcal{O}_F$ is a source of isomorphisms of HBAV:

Lemma 8.2. *Let S be any ring and let $(A, \iota, \lambda, \mu, \alpha)$ be a HBAV over S where μ is a μ_N-structure and α is either a $\Gamma_0(p^n)$, $\Gamma_1(p^n)$ or $\Gamma(p^n)$-level structure. Then for any $\eta \in \mathcal{O}_F^\times$, the map $\iota(\eta) : A \to A$ induces an isomorphism of HBAV*

$$\eta : (A, \iota, \eta^2\lambda, \eta^{-1}\mu_N, \eta\alpha) \xrightarrow{\sim} (A, \iota, \lambda, \mu_N, \alpha).$$

Here we write $\eta\alpha$ as a shorthand notation for the composition of the $\mathcal{O}_F$-linear map α with multiplication by η on either side of α. Similarly for $\eta^2\lambda$ and $\eta^{-1}\mu_N$.

Proof. A morphism of HBAVs $(B, \iota', \lambda', \mu', \alpha') \to (A, \iota, \lambda, \mu, \alpha)$ is an $\mathcal{O}_F$-linear isogeny $\varphi : B \to A$ making the following diagrams commute:

$$\begin{array}{ccc}
B^\vee & \xleftarrow{\varphi^\vee} & A^\vee \\
\uparrow{\scriptstyle \lambda'} & & \uparrow{\scriptstyle \lambda} \\
B \otimes \mathfrak{c} & \xrightarrow{\varphi} & A \otimes \mathfrak{c}
\end{array}
\qquad
\begin{array}{ccc}
\mathfrak{d}^{-1} \otimes \mu_N & = & \mathfrak{d}^{-1} \otimes \mu_N \\
\downarrow{\scriptstyle \mu'_N} & & \downarrow{\scriptstyle \mu_N} \\
B & \xrightarrow{\varphi} & A
\end{array}
\qquad
\begin{array}{ccc}
(\mathcal{O}/p^n)^2 & = & (\mathcal{O}/p^n)^2 \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\
B^\vee[p^n] & \xleftarrow{\varphi^\vee} & A^\vee[p^n].
\end{array}$$

Setting $B = A$, and $\varphi = [\eta]$, we see that $\lambda' = [\eta]^\vee \circ \lambda \circ [\eta] = \eta^2\lambda$, where in the second step we have used that $[\eta]^\vee \circ \lambda = \lambda \circ [\eta]$ since ι is stable under the Rosati involution. The second diagram implies $\mu'_N = [\eta]^{-1} \circ \mu_N = \eta^{-1}\mu_N$. The third diagram implies $\alpha' = [\eta]^\vee \circ \alpha = \eta\alpha$. □

Definition 8.3. The polarisation action of $\mathcal{O}_F^{\times,+}$ on X is given by letting $\eta \in \mathcal{O}_F^{\times,+}$ act via

$$\eta \cdot_{\mathrm{pol}} (A, \iota, \lambda, \mu_N) = (A, \iota, \eta\lambda, \mu_N).$$

As a consequence of Lemma 8.2 we see:

Proposition 8.4. [AIP16a, Lem. 8.1]. *This action of $\mathcal{O}_F^{\times,+}$ on X factors through the finite group $\Delta(N) := \mathcal{O}_F^{\times,+}/(1+N\mathcal{O}_F)^{\times 2}$, and makes $X \to X_G$ into a finite étale $\Delta(N)$-torsor.*

8.1. **The anti-canonical tower for G.** Our next goal is to construct from the anti-canonical tower of G^* the anti-canonical tower for G. For this, we move on from schemes to adic spaces.

Recall that, for either G or G^*, the datum of a $\Gamma_0(p^n)$-level structure is a the choice of an $\mathcal{O}_F$-submodule $C \subseteq A[p^n]$ etale locally isomorphic to $\mathcal{O}_F/p^n\mathcal{O}_F$. If necessary we will denote this level structure by $\Gamma_0(p^n)$ and $\Gamma_0^*(p^n)$ on G and G^* respectively. Similarly, on both G^* and G the notions of anticanonical level structures coincide. We thus obtain commutative diagrams

$$\begin{array}{ccc} X_{\Gamma_0^*(p^n)} & \longrightarrow & X_{G,\Gamma_0(p^n)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_G \end{array} \qquad \begin{array}{ccc} \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}_{G,\Gamma_0(p^n)}(\epsilon)_a \\ \downarrow & & \downarrow \\ \mathcal{X}(\epsilon) & \longrightarrow & \mathcal{X}_G(\epsilon) \end{array}$$

Lemma 8.5. *The above diagrams are both Cartesian. In particular, the morphism of adic spaces $\mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a \to \mathcal{X}_{G,\Gamma_0(p^n)}(\epsilon)_a$ is a finite étale $\Delta(N)$-torsor.*

Proof. The points of $X_{\Gamma_0^*(p^n)}$ correspond to HBAV $(A, \iota, \lambda, \mu_N, D)$ where $D \subseteq A[p^n]$ is a subgroup étale locally isomorphic to $\mathcal{O}/p^n$. For any $\eta \in \mathcal{O}_F^\times$, as D is an $\mathcal{O}_F$-module, we have $\eta D = D$, and hence the isomorphism of HBAVs induced by η from Lem. 8.2 identifies the tuples $(A, \iota, \eta^2\lambda, \eta^{-1} \cdot \mu_N, D)$ and $(A, \iota, \lambda, \mu_N, D)$ in $X_{\Gamma_0^*(p^n)}$. Thus, exactly as in Prop. 8.4, the polarisation action factors through $\Delta(N)$. It follows on the level of relative moduli descriptions that the top map in the above diagram is a $\Delta(N)$-torsor. This shows that the left diagram is Cartesian.

The case of the right hand side follows after adic analytification and restriction. □

By the following lemma, this allows us to pass to tilde-limits in the diagram on the right:

Lemma 8.6. *Let Δ be a finite group. Let $(\mathcal{X}_n \to \mathcal{Y}_n)_{n\in\mathbb{N}}$ be an inverse system of étale Δ-torsors of adic spaces over L. Suppose there is a perfectoid tilde-limit $\mathcal{X}_\infty \sim \varprojlim \mathcal{X}_n$. We moreover impose the technical condition that there is a cover of $\mathcal{Y}_0$ by open affinoid spaces V_0, with affinoid pullbacks $U_n := \mathrm{Spa}(R_n, R_n^+)$ to $\mathcal{X}_n$, such that R_∞ is affinoid perfectoid and $\varinjlim_n R_n \to R_\infty$ has dense image. Then $\mathcal{X}_\infty/\Delta =: \mathcal{Y}_\infty$ is perfectoid, $\mathcal{X}_\infty \to \mathcal{Y}_\infty$ is an étale Δ-torsor, and $\mathcal{Y}_\infty \sim \varprojlim \mathcal{Y}_n$.*

This is a special case of the statement of [She17, Cor. 2.3.5]. We focus on this special case since it suffices for our applications, and has the following simple proof:

Proof. Since the conclusions are local, by restricting to V_0 we are immediately reduced to the case that all $\mathcal{X}_n = \mathrm{Spa}(R_n, R_n^+)$ are affinoid, $\mathcal{X}_\infty = \mathrm{Spa}(R_\infty, R_\infty^+)$ is affinoid perfectoid and $\varinjlim R_n \to R_\infty$ has dense image. Then $\mathcal{Y}_n = \mathrm{Spa}(R_n^\Delta, R_n^{+,\Delta})$. It follows from [Han, Thm. 1.4] that $\mathcal{Y}_\infty := \mathcal{X}_\infty/\Delta = \mathrm{Spa}(R_\infty^\Delta, R_\infty^{+,\Delta})$ is perfectoid. Note that the assumptions are satisfied because we work over the perfectoid field L over $\mathbb{Q}_p$.

We claim that $\varinjlim_n R_n^\Delta \to R_\infty^\Delta$ has dense image. To see this, let $r \in R_\infty^\Delta$ and let $r' := r/|\Delta|$. Then we can find $r_n' \in R_n$ such that $r_n' \to r'$ inside R_∞. Let now $r_n := \sum_{g\in\Delta} gr_n'$, then clearly $r_n \in R_n^\Delta$. Since the Δ-action is continuous, and $r \in R_\infty^\Delta$, we then have $r_n \to \sum_{g\in\Delta} gr' = \frac{1}{|\Delta|}\sum_{g\in\Delta} gr = r$, as desired. This shows that $\mathcal{Y}_\infty \sim \varprojlim \mathcal{Y}_n$, since the condition on topological spaces follows from $|\mathcal{Y}_\infty| = |\mathcal{X}_\infty|/\Delta$.

That $\mathcal{X}_\infty \to \mathcal{Y}_\infty$ is a Δ-torsor now follows from the Cartesian diagrams expressing that $\mathcal{X}_n \to \mathcal{Y}_n$ is a Δ-torsor, by commuting product and perfectoid tilde-limit. □

Proposition 8.7. *There exists a perfectoid tilde-limit*

$$\mathcal{X}_{G,\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim \mathcal{X}_{G,\Gamma_0(p^n)}(\epsilon)_a.$$

Moreover, the natural map $\mathcal{X}_{\Gamma_0^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{G,\Gamma_0(p^\infty)}(\epsilon)_a$ is a finite étale $\Delta(N)$-torsor.

Proof. We apply Lem. 8.6 to the system of $\Delta(N)$-torsors $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}_{G,\Gamma_0(p^n)}(\epsilon)_a$. To see that the technical condition is satisfied, we note that by construction in Prop. 5.11, and by [SW13, Prop. 2.4.2], any affine open formal subscheme of the formal model $\mathfrak{X}^*(\epsilon)$ of $\mathcal{X}^*(\epsilon)$ pulls back to opens $\mathrm{Spa}(R_n, R_n^+) \subseteq \mathcal{X}_{\Gamma_0^*(p^n)}(\epsilon)_a$ for all $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that R_∞ is affinoid perfectoid and $\varinjlim R_n \to R_\infty$ has dense image. Thus also for any rational subset $\mathrm{Spa}(R_0, R_0^+)(\frac{T}{s})$, the map $\varinjlim R_n\langle\frac{T}{s}\rangle \to R_\infty\langle\frac{T}{s}\rangle$ has dense image. We conclude that any affinoid cover of $\mathcal{X}_G$ that is a rational refinement of an affine formal cover of $\mathfrak{X}_G^*(\epsilon) := \mathfrak{X}^*(\epsilon)/\Delta(N)$ is of the desired form. □

8.2. **Wild full level structures.** As before, we ultimately want to work with the full level structures $\Gamma(p^n)$ for G defined in §5.1.1, since this will provide the appropriate universal covering spaces we need to define overconvergent Hilbert modular forms. In this case, though, there are differences between $\Gamma(p^n)$ and $\Gamma^*(p^n)$-level structures which introduce new subtleties.

There is a space $\mathcal{X}_{G,\Gamma(p^n)} \to \mathcal{X}_{G,\Gamma_0(p)}$ which relatively represents a choice of an isomorphism $\alpha_n : (\mathcal{O}/p^n)^2 \to A^\vee[p^n]$ such that $\alpha_n(1,0)$ generates the corresponding subgroup for the $\Gamma_0(p^n)$-level, and there is a natural map $\mathcal{X}_{\Gamma^*(p^n)} \to \mathcal{X}_{G,\Gamma(p^n)}$. However, this map is not a torsor, as it is not surjective. In addition, $\mathcal{O}_F^{\times,+}$ no longer admits a polarisation action on $\mathcal{X}_{\Gamma^*(p^n)}$, due to the restrictive additional conditions on $\Gamma^*(p^n)$-level structures: indeed, changing λ changes the isomorphism b in diagram (5.2), and in general true this will not result in a similitude of pairings.

8.2.1. *'Hybrid' full level structures.* Bearing all of the above in mind, it is convenient to also introduce an intermediate space $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}$, relatively representing a choice of $\Gamma(p^n)$-level structure α_n over the Shimura variety $\mathcal{X}$ for G^*. This fits into a diagram

$$\begin{array}{ccccc} \mathcal{X}_{\Gamma^*(p^n)} & \xrightarrow{\beta_1} & \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\beta_2} & \mathcal{X}_{G,\Gamma(p^n)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & = & \mathcal{X} & \longrightarrow & \mathcal{X}_G. \end{array} \tag{8.1}$$

The polarisation action (8.3) now gives a well-defined action on this 'hybrid' space $\mathcal{X}_{\Gamma(p^n)}$, since the $\Gamma(p^n)$-level structures require no Weil pairing compatibility.

We also have a second natural left-action on this space:

Definition 8.8. The level structure (LS) action of $G(\mathbb{Z}/p^n\mathbb{Z}) = \mathrm{GL}_2(\mathcal{O}_F/p^n\mathcal{O}_F)$ on $\mathcal{X}_{\Gamma(p^n)}$ is given by letting $\gamma \in G(\mathbb{Z}/p^n\mathbb{Z})$ act as

$$\gamma \cdot_{\mathrm{LS}} (A, \iota, \lambda, \mu_N, \alpha_n) \mapsto (A, \iota, \lambda, \mu_N, \alpha_n \circ \gamma^\vee), \quad \gamma^\vee = \det(\gamma)\gamma^{-1}.$$

8.2.2. *Components and the $\mathcal{O}_F$-linear Weil pairing.* To understand the map $\mathcal{X}_{\Gamma^*(p^n)} \to \mathcal{X}_{G,\Gamma(p^n)}$, we begin by analysing β_1. For this, we require a description of the components of $\mathcal{X}_{\Gamma(p^n)}$ through the $\mathcal{O}_F$-linear Weil pairing $\widetilde{\mathbf{e}}_n$ (Defn. 5.6): Suppose (A, ι, λ) is a $\mathfrak{c}$-HBAV over S with $\Gamma(p^n)$-level structure $\alpha_n : (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \xrightarrow{\sim} A^\vee[p^n]$. Then $\widetilde{\mathbf{e}}_n$ induces an $\mathcal{O}_F$-linear isomorphism

$$\begin{aligned} \mathcal{O}_F/p^n\mathcal{O}_F \otimes \mathfrak{c}^{-1} =& (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \wedge (\mathcal{O}_F/p^n\mathcal{O}_F \otimes \mathfrak{c}^{-1})^2 \\ &\xrightarrow{\alpha_n \wedge (\alpha_n \otimes \mathfrak{c}^{-1})} A^\vee[p^n] \wedge A^\vee[p^n] \otimes \mathfrak{c}^{-1} \xrightarrow{(1, \lambda^{-1} \otimes \mathrm{id})} A^\vee[p^n] \wedge A[p^n] \xrightarrow{\widetilde{\mathbf{e}}_n} \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}. \end{aligned} \tag{8.2}$$

Equivalently, by tensoring with $\mathfrak{c}$, this can be described as a generator of the $\mathcal{O}_F$-module scheme $\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}$. We denote the subscheme of generators by $(\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n})^\times$.

In the universal situation over $\mathcal{X}_{\Gamma(p^n)}$, we conclude that the Weil pairing gives rise to a map

$$\mathbf{e}_{n,\beta} : \mathcal{X}_{\Gamma(p^n)} \to (\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n})^\times \xrightarrow{\beta^{-1}} (\mathcal{O}_F/p^n\mathcal{O}_F)^\times. \tag{8.3}$$

where we recall that $\beta \in \mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n}$ is the isomorphism chosen in Defn. 5.6.

Similarly for G we get a map $\mathbf{e}_n : \mathcal{X}_{G,\Gamma(p^n)} \to (\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n})^\times$. Next, we record two equivariance properties of the linearised Weil pairing.

Lemma 8.9. *(1) For $\gamma \in G(\mathbb{Z}/p^n\mathbb{Z})$, the action on $\mathcal{X}_{\Gamma(p^n)}$ fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\mathbf{e}_{n,\beta}} & (\mathcal{O}_F/p^n\mathcal{O}_F)^\times \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \det(\gamma)} \\ \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\mathbf{e}_{n,\beta}} & (\mathcal{O}_F/p^n\mathcal{O}_F)^\times. \end{array}$$

(2) For $\eta \in \mathcal{O}_F^{\times,+}$, the polarisation action on $\mathcal{X}_{\Gamma(p^n)}$ fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\mathbf{e}_{n,\beta}} & (\mathcal{O}_F/p^n\mathcal{O}_F)^\times \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta^{-1}} \\ \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\mathbf{e}_{n,\beta}} & (\mathcal{O}_F/p^n\mathcal{O}_F)^\times. \end{array}$$

Proof. Recall that γ acts by sending α_n to $\alpha_n \circ \gamma^\vee$. From (8.2) it is then clear that $\mathbf{e}_{n,\beta} \circ \gamma = \det(\gamma^\vee)\mathbf{e}_{n,\beta}$. The first part then follows from $\det \gamma^\vee = \det \gamma$. The second part holds since replacing $\lambda \to \eta \circ \lambda$ multiplies (8.2) by η^{-1} due to the λ^{-1} appearing. □

For any $c \in (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$, the fibre of $\mathbf{e}_{n,\beta}$ over c gives a component of $\mathcal{X}_{\Gamma(p^n)}$. This will be a connected component, as we shall discuss in the next section. Recall that by Defn. 5.4, the $\Gamma^*(p^n)$-level depends on $\beta \in c\mathfrak{d}^{-1}(1)$. We can now describe the space $\mathcal{X}_{\Gamma^*(p^n)}$ as follows:

Lemma 8.10. *The morphism $\mathcal{X}_{\Gamma^*(p^n)} \to \mathcal{X}_{\Gamma(p^n)}$ fits into a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}_{\Gamma^*(p^n)} & \xrightarrow{\mathbf{e}_{n,\beta}} & (\mathbb{Z}/p^n\mathbb{Z})^\times \\ \downarrow & & \downarrow \\ \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\mathbf{e}_{n,\beta}} & (\mathcal{O}_F/p^n\mathcal{O}_F)^\times \end{array}$$

Proof. We can check this on the level of schemes, where we can check on the level of moduli functors. Let $(A, \iota, \lambda, \mu_N, \alpha)$ be a HBAV over S corresponding to a point $x \in \mathcal{X}_{\Gamma^*(p^n)}(S)$. Then by (5.1), we recover the Weil pairing e_{p^n} from $\widetilde{\mathbf{e}}_{n,\beta}$ by composing with Tr. The level structure α therefore composes with λ^{-1} and $\widetilde{\mathbf{e}}_{n,\beta}$ to a pairing

$$(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \times (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \otimes \mathfrak{c}^{-1} \to \mathfrak{d}^{-1} \otimes \mu_{p^n} \xrightarrow{\mathrm{Tr}} \mu_{p^n}.$$

By definition, α is a $\Gamma_{\mathfrak{c}}^*(p^n)$-level structure if and only if this pairing is similar to the pairing

$$(\mathcal{O}_F/p^n\mathcal{O}_F)^2 \times (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \otimes \mathfrak{c}^{-1} \xrightarrow{\mathrm{id}\otimes\beta} (\mathcal{O}_F/p^n\mathcal{O}_F)^2 \times (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_{p^n})^2 \xrightarrow{\mathrm{Tr}} \mu_{p^n}.$$

After evaluation at 1 in the second factor, and tensoring with $\mathfrak{c}$, these each induce isomorphisms $\varphi_1, \varphi_2 : \mathcal{O}_F/p^n\mathcal{O}_F \to \mathfrak{c}\mathfrak{d}^{-1} \otimes \mu_{p^n}$. After composing with β^{-1}, the map φ_2 derived from the second pairing has image in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. The above pairings are now similar if and only if their ratio φ_1/φ_2^{-1} is in $\mathrm{Aut}(\mu_{p^n}) \subseteq \mathrm{Aut}(\mathfrak{c}\mathfrak{d}^{-1} \otimes \mu_{p^n})$, i.e. given by multiplication with $(\mathbb{Z}/p^n\mathbb{Z})^\times \subseteq (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$. Thus α is a $\Gamma^*(p^n)$-level structure if and only if $\mathbf{e}_{n,\beta}(x)$ is in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. □

8.2.3. *The map β_1.* Lems. 8.9 and 8.10 immediately imply that $\mathcal{X}_{\Gamma(p^n)}$ is a disjoint union of copies of $\mathcal{X}_{\Gamma^*(p^n)}$. More precisely, they imply the following corollary.

Corollary 8.11. *Let $(\mathcal{O}_F/p^n\mathcal{O}_F)^\times$ act on $\mathcal{X}_{\Gamma(p^n)}$ by letting η act via the level structure action of the matrix $\left(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The restriction of the action map to*

$$(\mathcal{O}_F/p^n\mathcal{O}_F)^\times \times \mathcal{X}_{\Gamma^*(p^n)} \to \mathcal{X}_{\Gamma(p^n)}$$

is then a $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor for the antidiagonal action and induces an isomorphism

$$\left[(\mathcal{O}_F/p^n\mathcal{O}_F)^\times \times \mathcal{X}_{\Gamma^*(p^n)}\right]/(\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathcal{X}_{\Gamma(p^n)}.$$

8.2.4. *The map β_2.* Next, we study the second map from (8.1), namely $\beta_2 : \mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{G,\Gamma(p^n)}$.

Lemma 8.12. *For any $\eta \in (1 + N\mathcal{O}_F)^\times$, set $\gamma := \left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta \end{smallmatrix}\right)$. Then the polarisation action of $\eta^2 \in \mathcal{O}_F^{\times,+}$ on $\mathcal{X}_{\Gamma(p^n)}$ coincides with the LS action on $\mathcal{X}_{\Gamma(p^n)}$ by γ^{-1}. In particular, if $\eta \in (1 + p^nN\mathcal{O}_F)^\times$, then the polarisation action of η^2 on $\mathcal{X}_{\Gamma(p^n)}$ is trivial.*

Proof. If $\eta \in (1+N\mathcal{O}_F)^\times$, then $\eta^{-1} \cdot \mu_N = \mu_N$. We note that acting on the polarisation via η^2 and then composing with the LS action by $\gamma = \gamma^\vee$ sends an HBAV $(A, \iota, \lambda, \mu_N, \alpha)$ to $(A, \iota, \eta^2\lambda, \mu_N, \eta\alpha)$

which is isomorphic to $(A, \iota, \lambda, \mu_N, \alpha)$ by Lem. 8.2, giving the first statement. The second is immediate, since if $\eta \in (1 + p^n N\mathcal{O}_F)^\times$, then η acts trivially on the level structure. □

Definition 8.13. For the tame levels we consider, the space $\mathcal{X}$ is connected, but this is no longer true of the spaces $\mathcal{X}_{\Gamma(p^n)}$ and $\mathcal{X}_{G,\Gamma(p^n)}$ for $n \geq 1$. We denote by $\mathcal{X}^0_{\Gamma(p^n)}$ and $\mathcal{X}^0_{G,\Gamma(p^n)}$ the respective identity components.

Definition 8.14. Let U_n be the cokernel in the exact sequence

$$1 \to (1 + p^n\mathcal{O}_F)^{\times,+} \to \mathcal{O}_F^{\times,+} \to (\mathcal{O}/p^n\mathcal{O})^\times \to U_n \to 1.$$

Lemma 8.15. *Via the Weil pairing, the sets of connected components are:*

(1) $\pi_0(X_{\mathfrak{c},\Gamma(p^n)}) = (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$,
(2) $\pi_0(X_{G,\mathfrak{c},\Gamma(p^n)}) = U_n$.

Proof. It suffices to prove this for $L = \mathbb{Q}_p^{\mathrm{cyc}}$, and we may choose a $\mathbb{Q}$-linear embedding $L \hookrightarrow \mathbb{C}$. The $\mathbb{C}$-points of $X_{\mathfrak{c},\Gamma^*(p^n)}$ then admit a description as $G^*(\mathbb{Q})^+\backslash[G^*(\mathbb{A}_f) \times \mathcal{S}]/K^*$, where $K^* := K^*(p^n) \cap K_1^*(N) \subset G^*(\mathbb{A}_f)$ is an open compact level subgroup. For $\ell \nmid p$, our choice of tame level ensures that $\det(K_\ell^*) = \mathbb{Z}_\ell^\times$, whilst $K_p^* = \Gamma^*(p^n)$ has determinant $1 + p^n\mathbb{Z}_p$. By strong approximation, the determinant thus induces an isomorphism from the component group

$$G^*(\mathbb{Q})^+\backslash G^*(\mathbb{A}_f)/K^* \xrightarrow{\sim} \hat{\mathbb{Z}}^\times/(1 + p^n\hat{\mathbb{Z}})^\times \cong (\mathbb{Z}/p^n\mathbb{Z})^\times.$$

Thus $\pi_0(X_{\mathfrak{c},\Gamma^*(p^n)}(\mathbb{C})) = (\mathbb{Z}/p^n\mathbb{Z})^\times$, which implies $\pi_0(X_{\mathfrak{c},\Gamma(p^n)}(\mathbb{C})) = (\mathcal{O}_F/p^n\mathcal{O}_F)^\times$ by Lem. 8.11. Similarly, for G we have $K := K(p^n) \cap K_1(N) \subset G(\mathbb{A}_f)$ and $G(\mathbb{Q})^+\backslash G(\mathbb{A}_f)/K$ equals

$$\mathrm{GL}_2(\mathcal{O}_F)^+\backslash \mathrm{GL}_2(\hat{\mathbb{Z}} \otimes \mathcal{O}_F)/K(p^n) \cap K_1(N) \xrightarrow{\sim} \mathcal{O}_F^{\times,+}\backslash(\hat{\mathbb{Z}} \otimes \mathcal{O}_F)^\times/(1 + p^n\hat{\mathbb{Z}} \otimes \mathcal{O}_F)^\times.$$

This is the strict ray class group of conductor p^n, which is an extension of $\mathrm{Cl}^+(\mathcal{O}_F)$ by U_n. After taking the fibre of $[\mathfrak{c}] \in \mathrm{Cl}^+(\mathcal{O}_F)$, this equals U_n as desired. □

Lemma 8.16.

(1) The map $\beta_2 : \mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{G,\Gamma(p^n)}$ *is a torsor for* $\Delta(p^n N) := \mathcal{O}_F^{\times,+}/(1 + p^n N\mathcal{O}_F)^{\times 2}$.
(2) The map $\beta_2 : \mathcal{X}^0_{\Gamma(p^n)} \to \mathcal{X}^0_{G,\Gamma(p^n)}$ *is a torsor for* $\Delta_n(N) := (1 + p^n\mathcal{O}_F)^{\times,+}/(1 + p^n N\mathcal{O}_F)^{\times 2}$.

Proof. Setup and notation like in the last proof, it suffices to see this for $X_{\Gamma(p^n)}(\mathbb{C}) \to X_{G,\Gamma(p^n)}(\mathbb{C})$. We first see from Lem. 8.15 that β_2 on connected components is the quotient $(\mathcal{O}_F/p^n\mathcal{O}_F)^\times \to (\mathcal{O}_F/p^n\mathcal{O}_F)^\times/(\mathcal{O}_F^{\times,+}/(1 + p^n\mathcal{O}_F)^{\times,+})$. It therefore suffices to prove that on identity components,

$$X^0_{\Gamma^*(p^n)}(\mathbb{C}) = X^0_{\Gamma(p^n)}(\mathbb{C}) \to X^0_{G,\Gamma(p^n)}(\mathbb{C})$$

is a torsor for the group $\Delta_n(N) := (1 + p^n\mathcal{O}_F)^{\times,+}/(1 + p^n N\mathcal{O}_F)^{\times,2}$. This map is the cover

$$\mathcal{G}^*\backslash\mathcal{H}^g \to \mathcal{G}\backslash\mathcal{H}^g \tag{8.4}$$

where $\mathcal{G}^* = K^* \cap G^*(\mathbb{Q})^+$ where $K^* = K^*(p^n) \cap K_1^*(N)$, and analogously for G. Recall that the kernel for the action of $G(\mathbb{Q})^+$ on $\mathcal{H}^g$ are the scalar matrices. Denoting by $\mathrm{P}\mathcal{G}$ the quotient of $\mathcal{G}$ by scalar matrices in $\mathcal{G}$. We note that the only scalar matrix in $\mathcal{G}^*$ is the identity. We therefore have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 1 & \longrightarrow & (1 + p^n N\mathcal{O}_F)^\times & \xrightarrow{x \mapsto x^2} & (1 + p^n N\mathcal{O}_F)^{\times,2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{G}^* & \longrightarrow & \mathcal{G} & \xrightarrow{\det} & (1 + p^n\mathcal{O}_F)^{\times,+} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{G}^* & \longrightarrow & \mathrm{P}\mathcal{G} & \longrightarrow & \Delta_n(N) & \longrightarrow & 1.
\end{array}$$

The bottom row tells us that the Galois group of the cover (8.4) is $\Delta_n(N)$, as desired. □

8.3. **Torsors over tame level.** To define overconvergent modular forms, we also need to understand the torsor structures obtained as we vary the wild level.

Definition 8.17. Let $m \leq n$.

(1) Let $\overline{\Gamma}_0(p^m,p^n) \subset G(\mathbb{Z}/p^n\mathbb{Z})$ and $\overline{\Gamma}_0^*(p^m,p^n) \subset G^*(\mathbb{Z}/p^n\mathbb{Z})$ denote the subgroups of matrices of the form $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$ with $p^m|c$.
(2) Let $Z_n := (1+N\mathcal{O}_F)^\times/(1+p^nN\mathcal{O}_F)^\times$, embedded diagonally into $\overline{\Gamma}_0(p^m,p^n)$.
(3) Let $\overline{\mathrm{P}\Gamma}_0(p^m,p^n) := \overline{\Gamma}_0(p^m,p^n)/Z_n$ be the quotient group.

By Lems. 8.9 and 8.10, the level structure action of $\Gamma(p^n)$ on $\mathcal{X}_{\Gamma(p^n)}$ restricts to an action of $\Gamma^*(p^n)$ on $\mathcal{X}_{\Gamma^*(p^n)}$. We then have:

Proposition 8.18. *Via actions on the level structure:*

(1) $\mathcal{X}_{\Gamma^*(p^n)} \to \mathcal{X}_{\Gamma_0^*(p^m)}$ *is a finite étale torsor for the group* $\overline{\Gamma}_0^*(p^m,p^n)$,

(2) $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{\Gamma_0^*(p^m)}$ *is a finite étale torsor for the group* $\overline{\Gamma}_0(p^m,p^n)$,

(3) $\mathcal{X}_{G,\Gamma(p^n)} \to \mathcal{X}_{G,\Gamma_0(p^m)}$ *is a finite étale torsor for the group* $\overline{\mathrm{P}\Gamma}_0(p^m,p^n)$.

Proof. Parts (1) and (2) follow from the moduli description. Part (3) follows from Lem. 8.16.1 and Lem. 8.19 below: the proof only uses the left hand side of (8.5), so this is not circular. □

8.3.1. *The diagonal torsor.* We now have a commutative diagram of towers of finite étale torsors

$$\begin{array}{ccc} \mathcal{X}_{\Gamma(p^n)} & \xrightarrow{\Delta(p^nN)} & \mathcal{X}_{G,\Gamma(p^n)} \\ {\scriptstyle \overline{\Gamma}_0(p^m,p^n)}\downarrow & \swarrow & \downarrow{\scriptstyle \overline{\mathrm{P}\Gamma}_0(p^m,p^n)} \\ \mathcal{X}_{\Gamma_0(p^m)} & \xrightarrow[\Delta(N)]{} & \mathcal{X}_{G,\Gamma_0(p^m)} \end{array} \tag{8.5}$$

Next, we describe the diagonal map in the above diagram, which should be a torsor for some group $\overline{E}(p^m,p^n)$ which can be described as an extension in two ways:

$$0 \to \overline{\Gamma}_0(p^m,p^n) \to \overline{E}(p^m,p^n) \to \Delta(N) \to 0,$$
$$0 \to \Delta(p^nN) \to \overline{E}(p^m,p^n) \to \overline{\mathrm{P}\Gamma}_0(p^m,p^n) \to 0.$$

It transpires that both extensions are non-split, reflecting our earlier observation that there is no polarisation action by $\Delta(N)$ on $\mathcal{X}_{\Gamma(p^n)}$.

In order to describe $\overline{E}(p^m,p^n)$ and its action, recall from Lem. 8.12 that for any $\eta \in (1+N\mathcal{O}_F)^\times$, the polarisation action of η^2 coincides with the action on the level structure via $\left(\begin{smallmatrix} \eta^{-1} & 0 \\ 0 & \eta^{-1} \end{smallmatrix}\right)$. We conclude that the combined action of $\overline{\Gamma}_0(p^m,p^n) \times \mathcal{O}_F^{\times,+}$ is such that the subgroup

$$(1+N\mathcal{O}_F)^\times \hookrightarrow \overline{\Gamma}_0(p^m,p^n) \times \mathcal{O}_F^{\times,+}, \quad \eta \mapsto \left(\left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta \end{smallmatrix}\right), \eta^2\right)$$

acts trivially on $\mathcal{X}_{\Gamma(p^n)}$. We therefore obtain an action of the quotient

$$\overline{E}(p,p^n) := \left(\overline{\Gamma}_0(p^m,p^n) \times \mathcal{O}_F^{\times,+}\right)/(1+N\mathcal{O}_F)^\times.$$

We now obtain a short exact sequences as above: first, we clearly have a sequence

$$0 \to \overline{\Gamma}_0(p^m,p^n) \xrightarrow{\gamma\mapsto(\gamma,1)} \overline{E}(p^m,p^n) \xrightarrow{(\gamma,x)\mapsto x} \Delta(N) \to 0.$$

Second, projection to the first factor induces a natural map $\overline{E}(p^m,p^n) \to \overline{\mathrm{P}\Gamma}_0(p^m,p^n)$. From the snake lemma diagram

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & (1+p^nN\mathcal{O}_F)^\times & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (1+N\mathcal{O}_F)^\times & \longrightarrow & (1+N\mathcal{O}_F)^\times & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to \mathcal{O}_F^{\times,+} & \to & \mathcal{O}_F^{\times,+} \times \overline{\Gamma}_0(p^m,p^n) & \longrightarrow & \overline{\Gamma}_0(p^m,p^n) & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ \mathcal{O}_F^{\times,+} & \longrightarrow & \overline{E}(p^m,p^n) & \longrightarrow & \overline{\mathrm{P}\Gamma}_0(p^m,p^n) & \longrightarrow & 0 \end{array} \qquad (x \mapsto x^2)$$

we see the kernel of this map is $\Delta(p^nN)$ embedded into $\overline{E}(p^m,p^n)$ via $x \mapsto (1,x)$, from which the second exact sequence follows. This shows:

Lemma 8.19. *The map* $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{G,\Gamma_0(p^m)}$ *is an étale torsor for the group* $\overline{E}(p^m,p^n)$.

Proof. By definition in diagram (8.5), this map is the composition of an étale $\overline{\Gamma}_0(p^m,p^n)$-torsor with an étale $\Delta(N)$-torsor. It is therefore finite étale. From the fact that $\overline{E}(p^m,p^n)$ acts on $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{G,\Gamma_0(p^m)}$, it is clear that the diagram defining the torsor property commutes. One then verifies that the diagram is Cartesian by decomposing it into smaller Cartesian diagrams induced from the torsor properties of $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{\Gamma_0(p^m)}$ and $\mathcal{X}_{\Gamma_0(p^m)} \to \mathcal{X}_{G,\Gamma_0(p^m)}$. □

8.4. **Passing to infinite level.** The following proposition is proven over $\mathbb{C}_p$ by Xu Shen [She17, Thm. 3.3.9] in the much greater generality of Shimura varieties of abelian type. In our special case, the version over $\mathbb{Q}_p^{\mathrm{cyc}}$ is easy to deduce from our preparations. We first note:

Lemma 8.20. *The group $\Delta_\infty(N) = \varprojlim_n \Delta_n(N)$ is finite and $\Delta_\infty(N) = \Delta_n(N)$ for $n \gg 0$.*

Proof. There is a natural injective map, compatible for varying n,

$$\Delta_n(N) = (1+p^n\mathcal{O}_F)^{\times,+}/(1+p^nN\mathcal{O}_F)^{\times,2} \hookrightarrow \mathcal{O}_F^{\times,+}/(1+N\mathcal{O}_F)^{\times,2} = \Delta(N).$$

Since $\Delta(N)$ is finite, it follows that $\Delta_n(N)$ stabilises for $n \gg 0$. □

Proposition 8.21. *(1) There exist perfectoid spaces $\mathcal{X}_{\Gamma(p^\infty)}$ and $\mathcal{X}_{G,\Gamma(p^\infty)}$ such that*

$$\mathcal{X}_{\Gamma(p^\infty)} \sim \varprojlim \mathcal{X}_{\Gamma(p^n)} \quad \text{and} \quad \mathcal{X}_{G,\Gamma(p^\infty)} \sim \varprojlim \mathcal{X}_{G,\Gamma(p^n)}.$$

(2) There also exist perfectoid spaces $\mathcal{X}^0_{\Gamma(p^\infty)} \sim \varprojlim \mathcal{X}^0_{\Gamma(p^n)}$ and $\mathcal{X}^0_{G,\Gamma(p^\infty)} \sim \varprojlim \mathcal{X}^0_{G,\Gamma(p^n)}$.

(3) There is a natural morphism $\mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{X}_{G,\Gamma(p^\infty)}$ which is a pro-étale torsor for the profinite group $\Delta(p^\infty N) = \varprojlim_n \Delta(p^nN)$.

(4) When restricted to connected components of the identity, it is a finite étale torsor $\mathcal{X}^0_{\Gamma(p^\infty)} \to \mathcal{X}^0_{G,\Gamma(p^\infty)}$ for the finite group $\Delta_\infty(N) = \varprojlim_n \Delta_n(N)$.

Proof. By Theorem 5.11.4, there is a perfectoid space $\mathcal{X}_{\Gamma^*(p^\infty)} \sim \varprojlim \mathcal{X}_{\Gamma^*(p^n)}$. By Cor. 8.11 we have $\mathcal{X}_{\Gamma(p^n)} = \mathcal{X}_{\Gamma^*(p^n)} \times [(\mathcal{O}_F/p^n\mathcal{O}_F)^\times/(\mathbb{Z}/p^n\mathbb{Z})^\times]$ on the level of adic spaces, and thus $\mathcal{X}_{\Gamma(p^\infty)} := \mathcal{X}_{\Gamma^*(p^\infty)} \times \mathcal{O}_p^\times/\mathbb{Z}_p^\times \sim \varprojlim_n \mathcal{X}_{\Gamma^*(p^n)} \times [(\mathcal{O}_F/p^n\mathcal{O}_F)^\times/(\mathbb{Z}/p^n\mathbb{Z})^\times] = \varprojlim_n \mathcal{X}_{\Gamma(p^n)}$.

For (2) we note that $\mathcal{X}^0_{\Gamma(p^n)} = \mathcal{X}^0_{\Gamma^*(p^n)}$, and the existence of the perfectoid space $\mathcal{X}^0_{\Gamma^*(p^\infty)} \sim \varprojlim \mathcal{X}^0_{\Gamma^*(p^n)}$ follows from [She17, Cor. 3.3.4]. From this we obtain the perfectoid space $\mathcal{X}^0_{G,\Gamma(p^\infty)}$, using Lem. 8.6, Lem. 8.16.2 and the fact that $\Delta_\infty(N) = \Delta_n(N)$ for $n \gg 0$. This lemma also gives (4). We deduce the second part of (1) from the second part of (2) by [She17, Prop. 3.3.5].

Finally, (3) follows from Lem. 8.16.1 by as usual applying the fact that perfectoid tilde-limits commute with fibre products to the diagram defining the torsor property. □

8.4.1. *The action of $G(\mathbb{Q}_p)$.* Since each $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}$ is an étale $G(\mathbb{Z}/p^n\mathbb{Z}) = \mathrm{GL}_2(\mathcal{O}_F/p^n\mathcal{O}_F)$-torsor, it follows that $\mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{X}$ is a pro-étale $G(\mathbb{Z}_p) = \mathrm{GL}_2(\mathcal{O}_p)$-torsor. Here we recall that $\gamma \in \mathrm{GL}_2(\mathcal{O}_p)$ acts by precomposition with $\gamma^\vee = \det(\gamma)\gamma^{-1}$ on the level structure $\mathcal{O}_p^2 \xrightarrow{\sim} T_pA^\vee$.

We shall now for a moment include the dependence on the polarisation ideal $\mathfrak{c}$ into the notation because, as in the Siegel case, the $G(\mathbb{Z}_p)$-action extends naturally to a $G(\mathbb{Q}_p)$-action which, in our case, permutes the spaces $\mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}$ over the polarisation ideals $\mathfrak{c}$, as we shall now describe.

Lemma 8.22. *Let (A,ι,λ,μ_N) be a HBAV. Let $\mathfrak{a} \subseteq \mathcal{O}_F$ be an ideal coprime to N and let $D \subseteq A[\mathfrak{a}]$ be any $\mathcal{O}_F$-submodule scheme. Then there is a unique way to make the isogeny $\varphi : A \to B := A/D$ into a morphism of HBAVs $(A,\iota,\lambda,\mu_N) \to (B,\iota',\lambda',\mu'_N)$. If $D \cong \oplus_{i=1}^k \mathcal{O}_F/\mathfrak{b}_i$ and $\mathfrak{b} := \mathfrak{b}_1 \cdots \mathfrak{b}_k$, then λ' is the unique $\mathfrak{cb}$-polarisation making the following diagram commute:*

$$\begin{array}{ccc} A \otimes \mathfrak{c} & \xrightarrow{\lambda} & A^\vee \\ {\scriptstyle \varphi^D \otimes \mathfrak{c}}\uparrow & & \uparrow{\scriptstyle \varphi^\vee} \\ B \otimes \mathfrak{cb} & \xrightarrow{\lambda'} & B^\vee. \end{array} \tag{8.6}$$

Here φ^D is such that $B \otimes \mathfrak{b} \xrightarrow{\varphi^D} A \xrightarrow{\varphi} B$ is the natural map $B \otimes \mathfrak{b} \to B \otimes \mathcal{O}_F = B$.

Proof. Let ι' be the quotient action and let μ'_N be the composition of μ_N with $A \to A/D$. It remains to construct λ' and show that it is a Deligne–Pappas polarisation as described. We refer to [KL05, §1.9] for the construction if D is of the form $D = \mathcal{O}_F/\mathfrak{b}_i$. The general case follows by decomposing into a chain of isogenies of this form. □

Let now $\gamma \in G(\mathbb{Q}_p) = \mathrm{GL}_2(F_p)$. If γ is an element of the form $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ for some $x \in \mathcal{O}_p$, we will see that the action we now define sends $A \mapsto A/A[x] = A \otimes (x)^{-1}$. For general $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we may therefore after rescaling assume that $\gamma \in M_2(\mathcal{O}_p) \cap \mathrm{GL}_2(F_p)$.

We may regard γ as acting on $\mathcal{O}_p^2 \otimes \mathfrak{c}^{-1}$. In particular, via $\lambda^{-1} \circ \alpha : \mathcal{O}_p^2 \otimes \mathfrak{c}^{-1} \xrightarrow{\sim} T_p A$, the matrix γ acts $\mathcal{O}_F$-linearly on $T_p A$ and thus on $A[p^n]$ for all n. For $n \to \infty$, the kernel D of $\gamma : A[p^n] \to A[p^n]$ stabilises. The automorphism γ now sends $(A, \iota, \lambda, \mu_N, \alpha)$ to the HBAV $(B := A/D, \iota', \lambda', \mu_N', \alpha')$ from Lem. 8.22, where $\alpha' : \mathcal{O}_p^2 \xrightarrow{\sim} T_p B^\vee$ is determined as follows:

Lemma 8.23. *There is a unique $\alpha' : \mathcal{O}_p^2 \xrightarrow{\sim} T_p B^\vee$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_p^2 & \xrightarrow{\alpha} & T_p A^\vee \\ {\scriptstyle \gamma^\vee}\uparrow & & \uparrow{\scriptstyle \varphi^\vee} \\ \mathcal{O}_p^2 & \xrightarrow{\alpha'} & T_p B^\vee. \end{array}$$

Proof. By Lem. 8.22, it suffices to show that there is a unique dotted arrow making the diagram

$$\begin{array}{ccccc} \mathcal{O}_p^2 & \xrightarrow{\alpha} & T_p A \otimes \mathfrak{c} & \xrightarrow{\lambda} & T_p A^\vee \\ {\scriptstyle \gamma^\vee}\uparrow & & {\scriptstyle \varphi^D}\uparrow & & \uparrow{\scriptstyle \varphi^\vee} \\ \mathcal{O}_p^2 & \dashrightarrow & T_p B \otimes \mathfrak{cb} & \xrightarrow{\lambda'} & T_p B^\vee \end{array}$$

commutative. Since all arrows become isomorphisms upon inverting p, it suffices to show that the cokernels of $\gamma^\vee$ and φ^D are identified by α. As usual, one sees that the cokernel of φ^D is given by $A[\mathfrak{b}]/D$. Since $\gamma \circ \gamma^\vee = \left(\begin{smallmatrix} \det\gamma & 0 \\ 0 & \det\gamma \end{smallmatrix}\right)$, we have $\operatorname{coker}\gamma^\vee = \operatorname{coker}(\det\gamma)/\operatorname{coker}\gamma$. Let now n be large enough that p^n kills $\operatorname{coker}\gamma$, then the Tor-sequence for quotienting by p^n shows that α sends $\operatorname{coker}\gamma$ to $\operatorname{coker}(\gamma : T_p A \to T_p A) = (\ker\gamma : A[p^n] \to A[p^n]) = D$. Second, we have $\det(\gamma)\mathcal{O}_p = \mathfrak{b}\mathcal{O}_p$, and the same Tor-argument shows that α sends $\operatorname{coker}(\det\gamma)$ to $\ker(\det\gamma : A[p^n] \to A[p^n]) = A[\mathfrak{b}]$. This shows that α sends $\operatorname{coker}\gamma^\vee$ to $A[\mathfrak{b}]/D = \operatorname{coker}\varphi^D$. □

It is clear from this characterisation of γ and the contravariance of $-^\vee$ that this is compatible with the multiplication in $G(\mathbb{Q}_p)$, and thus defines an action as desired. We moreover note that for $\gamma \in G(\mathbb{Z}_p)$, we have $\varphi = \mathrm{id}$ and therefore the action thus defined coincides with the action by precomposition with $\gamma^\vee$. Thus the $G(\mathbb{Q}_p)$-action extends the $G(\mathbb{Z}_p)$-action defined earlier.

Lemma 8.24. *Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.*

(1) $\mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{X}_{\Gamma_0(p^n)}$ is a pro-étale torsor for the group $\Gamma_0(p^n) := \varprojlim_m \overline{\Gamma}_0(p^n, p^m)$.

(2) $\mathcal{X}_{G,\Gamma(p^\infty)} \to \mathcal{X}_{G,\Gamma_0(p^n)}$ is a pro-étale torsor for the group $\mathrm{P}\Gamma_0(p^n) := \varprojlim_m \overline{\mathrm{P}\Gamma}_0(p^n, p^m)$.

(3) $\mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{X}_{G,\Gamma_0(p^n)}$ is a pro-étale torsor for the group $E(p^n) := \varprojlim_m \overline{E}(p^n, p^m)$.

(Notice we have swapped n and m; this is for notational convenience later on).

Proof. The diagrams expressing the torsor property are Cartesian since the corresponding Cartesian diagrams at finite level are, and perfectoid tilde-limits commute with fibre products. □

In summary, taking the limit over diagram (8.5), we thus get a diagram of pro-étale torsors

$$\begin{array}{ccccc} \mathcal{X}_{\Gamma^*(p^\infty)} & \longrightarrow & \mathcal{X}_{\Gamma(p^\infty)} & \xrightarrow{\Delta(p^\infty N)} & \mathcal{X}_{G,\Gamma(p^\infty)} \\ & {\scriptstyle \Gamma_0^*(p^n)}\searrow & \downarrow{\scriptstyle \Gamma_0(p^n)} & \searrow{\scriptstyle E(p^n)} & \downarrow{\scriptstyle \mathrm{P}\Gamma_0(p^n)} \\ & & \mathcal{X}_{\Gamma_0(p^n)} & \xrightarrow[\Delta(N)]{} & \mathcal{X}_{G,\Gamma_0(p^n)} \end{array} \tag{8.7}$$

where $\Delta(p^\infty N) := \varprojlim_n \Delta(p^n N)$.

8.4.2. *Comparison to $\mathcal{X}_{\Gamma^*(p^\infty)}$.* Taking limits, the Weil pairing morphism (8.3) induces maps

$$\varprojlim \mathbf{e}_{n,\beta} =: \mathbf{e}_\beta : \mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{O}_p^\times \qquad \text{and} \qquad \varprojlim \mathbf{e}_n =: \mathbf{e} : \mathcal{X}_{G,\Gamma(p^\infty)} \to \mathfrak{cd}^{-1}(1)^\times,$$

where the targets refer to the associated profinite perfectoid groups. Moreover, in the limit we obtain a level action of $G(\mathbb{Z}_p)$. Through this we define a level structure action of $\eta \in \mathcal{O}_p^\times$ acting by $\left(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In the limit, Lem. 8.9 and Cor. 8.11 give the following two lemmas.

Lemma 8.25. *The anti-diagonal action of $\mathcal{O}_p^\times$ restricts to a $\mathbb{Z}_p^\times$-torsor*

$$\mathcal{X}_{\Gamma^*(p^\infty)} \times \mathcal{O}_p^\times \longrightarrow \mathcal{X}_{\Gamma(p^\infty)}.$$

Lemma 8.26. *For any $(\gamma, x) \in E(p)$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{X}_{\Gamma(p^\infty)} & \xrightarrow{\mathbf{e}_\beta} & \mathcal{O}_p^\times \\ \downarrow{\scriptstyle (\gamma,x)} & & \downarrow{\scriptstyle \det(\gamma)x^{-1}} \\ \mathcal{X}_{\Gamma(p^\infty)} & \xrightarrow{\mathbf{e}_\beta} & \mathcal{O}_p^\times. \end{array}$$

Remark 8.27. We note that if we take the diagram from Lem. 8.5 and fiber it with (8.7) we can see that everything restricts to the anticanonical locus, i.e., in the statements above, we can replace $\mathcal{X}_{\Gamma(p^\infty)}, \mathcal{X}_{G,\Gamma(p^\infty)}$, etc with $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a, \mathcal{X}_{G,\Gamma(p^\infty)}(\epsilon)_a$, etc.

8.5. **Hodge–Tate period maps.**

Lemma 8.28. *There exist Hodge–Tate period maps making the following diagram commute:*

$$\begin{array}{ccccc} \mathcal{X}_{\Gamma^*(p^\infty)} & \longrightarrow & \mathcal{X}_{\Gamma(p^\infty)} & \longrightarrow & \mathcal{X}_{G,\Gamma(p^\infty)} \\ \downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}} \\ \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1 & \longrightarrow & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1 & \longrightarrow & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1. \end{array}$$

The map $\mathcal{X}_{\Gamma(p^\infty)} \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1$ is invariant for the polarisation action of $\mathcal{O}_F^{\times,+}$ on $\mathcal{X}_{\Gamma(p^\infty)}$.

Proof. We may define a map π_{HT} on $\mathcal{X}_{\Gamma^*(p^\infty)} \times \mathcal{O}_p^\times$ by projection from the first factor. This map is $\mathbb{Z}_p^\times$-invariant for the antidiagonal action since the $\mathbb{Z}_p^\times$-action on $\Gamma^*(p^\infty)$ just amounts to a rescaling of the basis vectors, which leaves the relative position of the kernel of the Hodge–Tate morphism invariant. Using the $\mathbb{Z}_p^\times$-torsor property from Lem. 8.25 in the pro-étale site, we conclude that π_{HT} descends to the second vertical map in the following diagram.

$$\begin{array}{ccccc} \mathcal{X}_{\Gamma^*(p^\infty)} \times \mathcal{O}_p^\times & \xrightarrow{\mathbb{Z}_p^\times} & \mathcal{X}_{\Gamma(p^\infty)} & \xrightarrow{\Delta(p^\infty N)} & \mathcal{X}_{G,\Gamma(p^\infty)} \\ \downarrow & & \vdots & & \vdots \\ \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1 & = & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1 & = & \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1 \end{array}$$

Similarly, the polarisation action of $\Delta(p^\infty N)$ clearly leaves π_{HT} invariant since it does not change the level structure. The same descent argument gives the third vertical map.

For the last statement, we note that $\mathcal{X}_{\Gamma^*(p^\infty)} \to \mathrm{Res}_{\mathcal{O}_F|\mathbb{Z}}\,\mathbb{P}^1$ is invariant under the polarisation action by $\mathbb{Z}_p^\times \cap \mathcal{O}_F^{\times,+}$ since the polarisation does not feature in the Hodge–Tate sequence. The result then follows from the construction via the $\mathbb{Z}_p^\times$-torsor $\mathcal{X}_{\Gamma^*(p^\infty)} \times \mathcal{O}_p^\times \to \mathcal{X}_{\Gamma(p^\infty)}$. □

9. Arithmetic overconvergent Hilbert modular forms

In §8, we exhibited various pro-étale torsors of perfectoid Hilbert modular varieties over $\mathcal{X}_G$. In this section, we use these to define overconvergent modular forms for G and compare them to those for G^*. We do this by defining four different sheaves, each using a different torsor, and then give a chain of comparisons that relate them. This will show that our modular forms for G agree with the previous sheaves defined in [AIP16a] by descending the sheaves for G^*.

Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight. Recall from 6.2 that by composition with $\rho : \mathcal{W} \to \mathcal{W}^*$ this also defines a bounded smooth weight for G^* which by Prop. 6.3 we may interpret as a morphism of adic spaces $\kappa : B_r(\mathcal{O}_p^\times : 1) \times \mathcal{U} \to \hat{\mathbb{G}}_m$. It follows from Lem. 8.28 that Defn. 6.4 goes through also for the infinite level spaces $\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a$ and $\mathcal{X}_{G,\Gamma(p^\infty)}(\epsilon)_a$. In particular, this gives us invertible functions $\kappa(c\mathfrak{z}+d)$ on each of these spaces. We denote them by the same letter since by definition they are compatible via pull-back along $\mathcal{X}_{\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{G,\Gamma(p^\infty)}(\epsilon)_a$.

Definition 9.1. Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. In the following, all infinite level sheaves are tacitly pushed forward to the indicated finite level base.

(1) The sheaf $\omega^{\kappa,+}_{G^*,(G^*,\Gamma^*,n)}$ of *integral modular forms for* G^* *via* $\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}$ is

$$\big\{f \in \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a} \;\big|\; \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0^*(p^n)\big\}.$$

(2) The sheaf $\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}$ of *integral modular forms for* G^* *via* $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}$ is

$$\big\{f \in \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a} \;\big|\; \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(p^n)\big\}.$$

(3) The sheaf $\omega^{\kappa,+}_{G,(G^*,\Gamma,n)}$ of *integral modular forms for* G *via* $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}$ is

$$\big\{f \in \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a} \;\big|(\gamma,x)^* f = \kappa^{-1}(c\mathfrak{z}+d)w_\kappa(x)f \quad \text{for all } (\gamma,x) \in E(p^n) \quad\big\}.$$

(4) The sheaf $\omega^{\kappa,+}_{G,(G,\Gamma,n)}$ of *integral modular forms for* G *via* $\mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}$ is

$$\big\{f \in \mathcal{O}^+_{\mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a} \big| \quad \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)w_\kappa(\det\gamma)f \quad \text{for all } \gamma \in \mathrm{P}\Gamma_0(p^n) \quad\big\}.$$

We note that (1) is the sheaf of modular forms for G^* from Defn. 6.5, denoted there by ω^κ_n. We have switched to the notation above for clearer comparison to (2), (3) and (4). The goal of this secion is to relate these sheaves. More precisely:

(i) in Lem. 9.3 below, we will see that the sheaves (1) and (2) are isomorphic;
(ii) in Lem. 9.6, we will see that sheaf (3) is obtained from (2) by taking $\Delta(N)$-invariants;
(iii) in Lem. 9.7, we will see that the sheaves (3) and (4) are isomorphic.

Before we start giving comparison maps, however, we need to check:

Lemma 9.2. *The conditions in (3) and (4) above are well-defined; that is, they do not depend on the choice of representatives* (γ, x) *or* γ *respectively.*

Proof. The relation of w_κ and κ given in Def. 6.2 implies that

$$\kappa^{-1}(\eta)w_\kappa(\eta^2) = t_\kappa \circ N_{F/\mathbb{Q}}(\eta) = 1 \text{ for all } \eta \in \mathcal{O}_F^{\times,+}. \tag{9.1}$$

For (3), this implies that for any $(\gamma,x) \in \Gamma_0(p^n) \times \mathcal{O}_F^{\times,+}$, the factor $\kappa^{-1}(c\mathfrak{z}+d)w_\kappa(x)$ only depends on the image of (γ,x) in $E(p^n)$; indeed, for any $\eta \in \mathcal{O}_F^{\times,+}$ the translate $(\gamma \left(\begin{smallmatrix}\eta & 0\\ 0 & \eta\end{smallmatrix}\right), x\eta^2)$ has the same associated factor

$$\kappa^{-1}(c\eta\mathfrak{z}+d\eta)w_\kappa(x\eta^2) = \kappa^{-1}(\eta)w_\kappa(\eta^2)\kappa^{-1}(c\mathfrak{z}+d)w_\kappa(x) = \kappa^{-1}(c\mathfrak{z}+d)w_\kappa(x).$$

For (4), we similarly note that for any $\eta \in (1+N\mathcal{O}_F)^{\times,+}$, setting $\gamma = \left(\begin{smallmatrix}\eta & 0\\ 0 & \eta\end{smallmatrix}\right)$ results in the factor $\kappa^{-1}(\eta)w_\kappa(\eta^2) = 1$. By continuity, the same is true for γ in the topological closure Z_∞ of $(1+N\mathcal{O}_F)^{\times,+}$ in $\mathcal{O}_p^{\times,+}$. As $\mathrm{P}\Gamma_0(p^n) = \varprojlim_m \overline{\mathrm{P}\Gamma}_0(p^n,p^m) = \varprojlim_m \overline{\Gamma}_0(p^n,p^m)/Z_m = \Gamma_0(p^n)/Z_\infty$, this shows that for any $\gamma \in \Gamma_0(p^n)$, the factor $\kappa^{-1}(c\mathfrak{z}+d)w_\kappa(\det\gamma)$ only depends on the image of γ in $\mathrm{P}\Gamma_0(p^n)$. This shows that the condition in (4) is well-defined. □

Lemma 9.3. *Let* $\kappa : \mathcal{U} \to \mathcal{W}^*$ *be a smooth weight. The natural morphism of torsors* $\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$ *over* $\mathcal{X}_{\mathcal{U},\Gamma_0^*(p^n)}(\epsilon)_a$ *induces a natural isomorphism*

$$\omega^{\kappa,+}_{G^*,(G^*,\Gamma^*,n)} \cong \omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}.$$

In particular, the definition of forms for G^* *is independent of the choice of* Γ *or* Γ^*.

Proof. By Lem. 8.28, for any $\gamma \in \Gamma_0^*(p^n)$, the function $\kappa^{-1}(c\mathfrak{z}+d)$ on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$ pulls back to $\kappa^{-1}(c\mathfrak{z}+d)$ on $\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a$. Since the map $\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$ is equivariant under $\Gamma_0^*(p^n) \to \Gamma_0(p^n)$, it follows that the associated map $\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a} \to \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a}$ restricts to

$$\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)} \to \omega^{\kappa,+}_{G^*,(G^*,\Gamma^*,n)}. \tag{9.2}$$

To construct an inverse, let $f \in \omega^{\kappa,+}_{G^*,(G^*,\Gamma^*,n)}$. Note that f is invariant under the action of $\mathbb{Z}_p^\times$, since for any $\epsilon \in \mathbb{Z}_p^\times$, we have $\gamma_\epsilon = \left(\begin{smallmatrix}\epsilon & 0\\ 0 & 1\end{smallmatrix}\right)$ acting via $\gamma_\epsilon^* f = \kappa^{-1}(1)f = f$. Consider now

$$\mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a \longleftarrow \mathcal{X}_{\mathcal{U},\Gamma^*(p^\infty)}(\epsilon)_a \times \mathcal{O}_p^\times \longrightarrow \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a,$$

where the right hand morphism is the $\mathbb{Z}_p^\times$-torsor from Lem. 8.25 and the left hand morphism is simply the projection. Since f is $\mathbb{Z}_p^\times$-invariant, the pullback f' of f under the left map is

invariant under the antidiagonal $\mathbb{Z}_p^\times$-action. Since the morphism on the right is a $\mathbb{Z}_p^\times$-torsor of affinoid perfectoid spaces, this implies that f' descends uniquely down the right to a function f'' on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$. We claim that $f'' \in \omega^\kappa_{G^*,(G^*,\Gamma^*,n)}$, that is, it transforms as required under $\Gamma_0(p^n)$. This group is generated by $\left(\begin{smallmatrix}\mathcal{O}_p^\times & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\Gamma_0^*(p^n)$, so it suffices to check that

$$\gamma_\epsilon^* f'' = \kappa^{-1}(0\mathfrak{z}+1)f'' = f'' \text{ for all } \epsilon \in \mathcal{O}_p^\times, \tag{9.3}$$

$$\gamma^* f'' = \kappa^{-1}(c\mathfrak{z}+d)f'' \text{ for all } \gamma \in \Gamma_0^*(p^n). \tag{9.4}$$

As f'' came from pullback from the left, the function f' is invariant for the $\mathcal{O}_p^\times$-action, and thus the same is true for f'', where $\epsilon \in \mathcal{O}_p^\times$ acts via $\gamma_\epsilon = \left(\begin{smallmatrix}\epsilon & 0\\ 0 & 1\end{smallmatrix}\right)$. This gives (9.3). Also, since f was a modular form and the above maps are all equivariant for the natural $\Gamma_0^*(p^n)$-action, we also have (9.4). Thus $f'' \in \omega^\kappa_{G^*,(G^*,\Gamma^*,n)}$, and we see directly that $f \mapsto f''$ gives an inverse to (9.2). □

We now use étale descent along $\pi : \mathcal{X}_{\mathcal{U}}(\epsilon) \to \mathcal{X}_{G,\mathcal{U}}(\epsilon)$ to relate the sheaves for G^* and G. Explicitly, this can be done by endowing $\pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}$ with a $\Delta(N)$-action given by a twist of the $\mathcal{O}_F^{\times,+}$-action on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$, as follows:

Definition 9.4. Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a smooth bounded weight. We define a w_κ-twisted polarisation left-action of $\mathcal{O}_F^{\times,+}$ on $\pi_* q_* \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a}$ by letting $\varepsilon \in \mathcal{O}_F^{\times,+}$ act as

$$\varepsilon \cdot_{w_\kappa} f := w_\kappa(\varepsilon)\cdot(\varepsilon^{-1})^* f$$

where w_κ is as in 6.2 and the action on the right side is the polarisation action on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$.

Lemma 9.5. *The w_κ-twisted action of $\mathcal{O}_F^{\times,+}$ restricts to $\pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}$, where it factors through an action of $\Delta(N)$. Furthermore, this action coincides with the action defined by* [AIP16a].

Proof. It is clear from the moduli description that the polarisation action of $\mathcal{O}_F^{\times,+}$ and the level structure action of $\Gamma_0(p^n)$ commute on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$. By Lemma 8.28, the action moreover leaves π_{HT} and thus $\mathfrak{z}$ invariant. We therefore have for any $\varepsilon \in \mathcal{O}_F^{\times,+}$ and any $f \in \pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}$

$$\gamma^*(\varepsilon \cdot_{w_\kappa} f) = w_\kappa(\varepsilon)\varepsilon^{-1*}\gamma^* f = w_\kappa(\varepsilon)\varepsilon^{-1*}(\kappa^{-1}(c\mathfrak{z}+d)f) = \kappa^{-1}(c\mathfrak{z}+d)\varepsilon \cdot_{w_\kappa} f.$$

This shows that the action restricts. Next, if $\eta \in \mathcal{O}_F^{\times,+}$, then η^2 acts on $f \in \pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}$ via

$$\eta^2 \cdot_{w_\kappa} f = w_\kappa(\eta^2)\cdot(\eta^{-2})^* f \overset{\text{Lem. 8.12}}{=\!=} w_\kappa(\eta^2)\cdot\left(\begin{smallmatrix}\eta & 0\\ 0 & \eta\end{smallmatrix}\right)^* f = w_\kappa(\eta^2)\kappa^{-1}(\eta) f \overset{(9.1)}{=} f,$$

and in particular the subgroup $(1+N\mathcal{O}_F)^{\times 2}$ acts trivially; thus the action factors through $\Delta(N)$ as desired. The last statement follows from Lem. 9.3 and Thm. 7.14 since Defn. 9.4 and the definition in [AIP16b, Section 4.1] match up: here we use that the polarisation action commutes with $\mathfrak{s}$ since it leaves the wild level α and the Hodge–Tate morphism unchanged. □

Lemma 9.6. *We have an equality of subsheaves of $\mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a}$ on $\mathcal{X}_{G,\mathcal{U},\Gamma_0(p)}(\epsilon)_a$*

$$\omega^{\kappa,+}_{G,(G^*,\Gamma,n)} = (\pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)})^{\Delta(N)}.$$

Proof. Let $f \in \omega^{\kappa,+}_{G,(G^*,\Gamma,n)}$. By Lem. 9.2, the action of $E(p^n)$ that defines the modular forms property for f is induced by an action of $\Gamma_0(p^n)\times\mathcal{O}_F^{\times,+}$; from this, it is clear that

$$\gamma^* f = (\gamma,1)^* f = \kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(p^n),$$

and thus $f \in \pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)}$. To see that it is also $\Delta(N)$-equivariant, it suffices to note that

$$x^* f = (1,x)^* f = w_\kappa(x) f \text{ for all } x \in \mathcal{O}_F^{\times,+},$$

which implies that for the w_κ-twisted action of x we have $x \cdot_{w_\kappa} f = w_\kappa(x)(x^{-1})^* f = f$. Thus $f \in (\pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)})^{\Delta(N)}$ as desired. The converse follows by reversing the above calculations. □

To compare $\omega^{\kappa,+}_{G,(G^*,\Gamma,n)}$ to $\omega^{\kappa,+}_{G,(G,\Gamma,n)}$, pulling back functions along $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)} \to \mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}$ is not enough: We also need an additional twist that changes the factor of $\kappa^{-1}(c\mathfrak{z}+d)w_\kappa(x)$ in the definition of the former into the factor of $\kappa^{-1}(c\mathfrak{z}+d)w_\kappa(\det\gamma)$ used in the latter. To describe

this, let $\kappa : \mathcal{U} \to \mathcal{W} \to \mathcal{W}^*$ be a smooth weight and recall from the discussion before Lem. 8.26 the Weil pairing morphism $\mathbf{e}_\beta : \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)} \to \mathcal{O}_p^\times$. We use this to define a composite morphism

$$w_\kappa(\mathbf{e}_\beta) : \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)} \xrightarrow{\mathbf{e}_\beta} \mathcal{O}_p^\times \xrightarrow{w_\kappa} \hat{\mathbb{G}}_m.$$

Restricting to the subspace $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$, the universal property of $\hat{G}_m$ associates to $w_\kappa(\mathbf{e}_\beta)$ a function in $\mathcal{O}^+(\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a)^\times$ that we might reasonably also denote by $w_\kappa(\mathbf{e}_\beta)$. By Lem. 8.26,

$$(\gamma, x)^* w_\kappa(\mathbf{e}_\beta) = w_\kappa(x^{-1}) w_\kappa(\det\gamma) w_\kappa(\mathbf{e}_\beta) \quad \text{for any } (\gamma, x) \in E(p^n). \tag{9.5}$$

Lemma 9.7. *Let π_∞ be the natural morphism of torsors*

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a & \xrightarrow{\pi_\infty} & \mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \\ {\scriptstyle E(p^n)}\downarrow & & \downarrow{\scriptstyle \mathrm{P}\Gamma_0(p^n)} \\ \mathcal{X}_{G,\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a & = & \mathcal{X}_{G,\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a. \end{array}$$

Then sending $f \mapsto w_\kappa^{-1}(\mathbf{e}_\beta) \cdot \pi_\infty^* f$ defines an isomorphism of $\mathcal{O}^+$-modules on $\mathcal{X}_{G,\mathcal{U},\Gamma_0(p^n)}(\epsilon)_a$

$$\omega^{\kappa,+}_{G,(G,\Gamma,n)} \xrightarrow{\sim} \omega^{\kappa,+}_{G,(G^*,\Gamma,n)}.$$

Proof. Let $f \in \omega^{\kappa,+}_{G,(G,\Gamma,n)}$. For any $(\gamma, x) \in E(p^n)$, we have

$$\begin{aligned} (\gamma, x)^*(w_\kappa^{-1}(\mathbf{e}_\beta)\pi_\infty^* f) &= (\gamma, x)^* w_\kappa^{-1}(\mathbf{e}_\beta) \cdot (\gamma, x)^* \pi_\infty^* f \\ &= w_\kappa(x) w_\kappa^{-1}(\det\gamma) w_\kappa^{-1}(\mathbf{e}_\beta) \pi_\infty^* \gamma^* f \\ &= w_\kappa(x) w_\kappa^{-1}(\det\gamma) w_\kappa^{-1}(\mathbf{e}_\beta) \pi_\infty^* (\kappa^{-1}(c\mathfrak{z} + d) w_\kappa(\det\gamma) f) \\ &= \kappa^{-1}(c\mathfrak{z} + d) w_\kappa(x) w_\kappa^{-1}(\mathbf{e}_\beta) \pi_\infty^* f. \end{aligned}$$

In the second step we have used equation (9.5) and the fact that π_∞ is equivariant with respect to the projection $E(p^n) \to \mathrm{P}\Gamma_0(p^n)$, $(\gamma, x) \mapsto \gamma$, in the third step we use $f \in \omega^{\kappa,+}_{G,(G,\Gamma,n)}$, and in the last step we use that by Lem. 8.28, $\pi_\infty^*(\kappa^{-1}(c\mathfrak{z} + d)) = \kappa^{-1}(c\mathfrak{z} + d)$. This identity shows that indeed $w_\kappa^{-1}(\mathbf{e}_\beta)\pi_\infty^* f \in \omega^\kappa_{G,(G^*,\Gamma,n)}$, so the map in the statement of the proposition is well-defined.

To see that the map is an isomorphism, take now $f \in \omega^{\kappa,+}_{G,(G^*,\Gamma,n)}$ and consider the function $g = w_\kappa(\mathbf{e}_\beta) f$ on $\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$. We claim that g descends uniquely to a function on $\mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$. To see this, since $\pi_\infty : \mathcal{X}_{\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$ is a perfectoid $\Delta(p^\infty N)$-torsor, it suffices to show that g is $\Delta(p^\infty N)$-invariant. It suffices by continuity to show that it is invariant for the dense subgroup $\mathcal{O}_F^{\times,+} \hookrightarrow \Delta(p^\infty N)$. For this we calculate that for any $x \in \mathcal{O}_F^{\times,+}$ we have

$$x^* g = (1, x)^* g = (1, x)^*(w_\kappa(\mathbf{e}_\beta))(1, x)^* f = w_\kappa(x)^{-1} w_\kappa(\mathbf{e}_\beta) \kappa^{-1}(1) w_\kappa(x) f = w_\kappa(\mathbf{e}_\beta) f = g,$$

where we have again used equation (9.5). This shows that g indeed arises as pullback of a unique function $h = h(f)$ on $\mathcal{X}_{G,\mathcal{U},\Gamma(p^\infty)}(\epsilon)_a$ with $\pi_\infty^* h = g$. We claim that $h \in \omega^{\kappa,+}_{G,(G,\Gamma,n)}$. For this we have to check by definition that $\gamma^* h$ equals $\kappa^{-1}(c\mathfrak{z} + d) w_\kappa(\det\gamma) h$. The morphism of sheaves $\pi_\infty^* : \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma(p^\infty)(\epsilon)_a}} \to \mathcal{O}^+_{\mathcal{X}_{\mathcal{U},\Gamma_0(p^\infty)(\epsilon)_a}}$ is the pullback of a pro-étale map, hence injective; thus, as it is also $\Gamma_0(p^n)$-equivariant, it suffices to check this property on $\pi_\infty^* h = g$ with respect to the action of $\Gamma_0(p^n) \to E(p^n)$. But by definition of g as $w_\kappa(\mathbf{e}_\beta) f$, we have for any $\gamma \in \Gamma_0(p^n)$ that

$$\gamma^* g = (\gamma, 1)^*(w_\kappa(\mathbf{e}_\beta))(\gamma, 1)^* f = w_\kappa(\det(\gamma)) w_\kappa(\mathbf{e}_\beta) \kappa^{-1}(c\mathfrak{z} + d) f = \kappa^{-1}(c\mathfrak{z} + d) w_\kappa(\det(\gamma)) g.$$

This shows that g transforms under the action of $\Gamma_0(p^n)$ as desired, and thus so does h. This shows $h \in \omega^{\kappa,+}_{G,(G,\Gamma,n)}$ as desired. Moreover, by construction we have $f = w_\kappa^{-1}(\mathbf{e}_\beta)\pi_\infty^* h$.

This shows that the map in the proposition is injective and surjective, thus an isomorphism. □

Definition 9.8. Let $\kappa : \mathcal{U} \to \mathcal{W}$ be a bounded smooth weight and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For $n \neq 0$, let $\omega^{\kappa,+}_{G,\mathfrak{c},n}$ be the sheaf $\omega^{\kappa,+}_{G,(G,\Gamma,n)}$ from Defn. 9.1 (4), noting that we have reintroduced our fixed $\mathfrak{c}$ to the notation. As before, we define the rational version $\omega^\kappa_{G,\mathfrak{c},n}$ by replacing $\mathcal{O}^+_{\mathcal{X}}$ with $\mathcal{O}_{\mathcal{X}}$, or equivalently by inverting p. We use the Atkin–Lehner isomorphism to define '$n = 0$' versions

$$\omega^\kappa_{G,\mathfrak{c}} = \omega^\kappa_{G,\mathfrak{c},0} := \mathrm{AL}_* \omega^\kappa_{G,\mathfrak{c},1}, \quad \omega^{\kappa,+}_{G,\mathfrak{c}} = \omega^{\kappa,+}_{G,\mathfrak{c},0} := \mathrm{AL}_* \omega^{\kappa,+}_{G,\mathfrak{c},1}$$

which are sheaves on $\mathcal{X}_{G,\mathfrak{c},\mathcal{U}}(\epsilon)$. For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the space of *overconvergent arithmetic Hilbert modular forms of tame level μ_N, p-level $\Gamma_0(p^n)$, polarisation ideal $\mathfrak{c}$, radius of*

overconvergence ϵ and weight κ, and its integral subspace, are then

$$M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,\mathfrak{c}) := \mathrm{H}^0(\mathcal{X}_{G,\mathfrak{c},\mathcal{U}}(\epsilon),\omega^\kappa_{G,\mathfrak{c},n}),$$
$$M_\kappa^{G,+}(\Gamma_0(p^n),\mu_N,\epsilon,\mathfrak{c}) := \mathrm{H}^0(\mathcal{X}_{G,\mathfrak{c},\mathcal{U}}(\epsilon),\omega^{\kappa,+}_{G,\mathfrak{c},n}).$$

Remark 9.9. One can define cusp forms for G in much the same way as for G^* (Rem. 6.9).

As in [AIP16b, Lem. 4.5], the polarisation action by p-adic units $x \in F^{\times,+}$ defines isomorphisms

$$P_x : M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,\mathfrak{c}) \xrightarrow{\sim} M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,x\mathfrak{c})$$

We will see below that the action of the Hecke operators permutes the spaces $M_\kappa^G(\mu_N,\epsilon,\mathfrak{c})$ by scaling the polarisation ideal. To obtain a Hecke stable space, one therefore also defines:

Definition 9.10. Let $\mathfrak{R}_F^{(p)}$ be the group of fractional ideals of F coprime to p and $\mathfrak{T}_F^{+,(p)}$ be the positive elements which are p-adic units, then $\mathfrak{R}_F^{(p)}/\mathfrak{T}_F^{+,(p)}$ is the narrow class group. We let

$$M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon) := \left(\bigoplus_{\mathfrak{c}\in\mathfrak{R}_F^{(p)}} M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,\mathfrak{c})\right)\Big/\left\langle P_x(f)-f \text{ for all } x\in\mathfrak{T}_F^{+,(p)}\right\rangle.$$

This is the space of arithmetic overconvergent Hilbert modular forms of weight κ, tame level μ_N and radius of overconvergence ϵ. One can define integral versions of these spaces by using $\omega^{\kappa,+}_{G,\mathfrak{c},n}$.

Putting everything together, we obtain the desired comparison to the sheaf of Hilbert modular forms for G as defined by Andreatta–Iovita–Pilloni. For this we first recall the definition:

Definition 9.11. Let $\omega^{\kappa,+}_{G,\mathfrak{c},\mathrm{AIP,n}} := (\pi_*\omega^{\kappa,+}_{G^*,\mathfrak{c},\mathrm{AIP,n}})^{\Delta(N)}$. This is the integral analytic incarnation of the sheaf of Hilbert modular forms for G on $\mathcal{X}_{G,\mathfrak{c},\mathcal{U}}(\epsilon)$ denoted by $\omega^{\kappa^{\mathrm{un}}_G}$ in [AIP16a, §8.2].

Theorem 9.12. *Let $\kappa:\mathcal{U}\to\mathcal{W}$ be a smooth bounded weight, $n\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$ and $0\leq\epsilon\leq\epsilon_\kappa$. There is a natural isomorphism*

$$\omega^{\kappa,+}_{G,\mathfrak{c},n} \cong \omega^{\kappa,+}_{G,\mathfrak{c},\mathrm{AIP},n}$$

of $\mathcal{O}^+$-modules on $\mathcal{X}_{G,\mathfrak{c},\mathcal{U}}(\epsilon)$. In particular, $\omega^{\kappa,+}_{G,\mathfrak{c},n}$ is an invertible $\mathcal{O}^+$-module. By inverting p, it induces a Hecke-equivariant isomorphism of line bundles $\omega^\kappa_{G,\mathfrak{c},n} = \omega^\kappa_{G,\mathfrak{c},\mathrm{AIP},n}$.

Proof. To ease notation, we suppress the dependence on $\mathfrak{c}$. By Thm. 7.14, we have $\omega^{\kappa,+}_{G^*} = \omega^{\kappa,+}_{G^*,\mathrm{AIP}}$. By combining the lemmas of §9, we conclude that

$$\omega^{\kappa,+}_{G,n} \overset{(9.8)}{=} \omega^{\kappa,+}_{G,(G,\Gamma,n)} \overset{(9.7)}{=} \omega^{\kappa,+}_{G,(G^*,\Gamma,n)} \overset{(9.6)}{=} (\pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma,n)})^{\Delta(N)}$$
$$\overset{(9.3)}{=} (\pi_*\omega^{\kappa,+}_{G^*,(G^*,\Gamma^*,n)})^{\Delta(N)} \overset{(7.14)\&(9.5)}{=} (\pi_*\omega^{\kappa,+}_{G^*,\mathrm{AIP},n})^{\Delta(N)} \overset{(9.11)}{=} \omega^{\kappa,+}_{G,\mathrm{AIP},n}$$

as desired. We postpone the proof of Hecke-equivariance to §10. □

10. Hecke operators

Throughout this section, we fix a smooth bounded weight $\kappa:\mathcal{U}\to\mathcal{W}\to\mathcal{W}^*$ and $0\leq\epsilon\leq\epsilon_\kappa$. To ease notation, in this section we suppress the subscript $\mathcal{U}$ from our Hilbert modular varieties.

10.1. **The tame Hecke operators.** Let $\mathfrak{a}\subseteq\mathcal{O}_F$ be any ideal coprime to $\mathfrak{n}$ and p. Let us denote by $\mathcal{X}_{\mathfrak{c},\Gamma_0(\mathfrak{a})}(\epsilon)$ the Hilbert modular variety of tame level $\mu_N\cap\Gamma_0(\mathfrak{a})$, representing tuples $(A,\iota,\lambda,\mu_N,D)$ where $D\subseteq A[\mathfrak{a}]$ is a closed $\mathcal{O}_F$-submodule scheme that is étale locally isomorphic to $\mathcal{O}_F/\mathfrak{a}$, and $\lambda: A\otimes\mathfrak{c}\xrightarrow{\sim}A^\vee$. Let us denote by $\pi_1:\mathcal{X}_{\mathfrak{c},\Gamma_0(\mathfrak{a})}(\epsilon)\to\mathcal{X}_{\mathfrak{c}}(\epsilon)$ the forgetful morphism. By Lem. 8.22, there is a second map defined by sending $A\mapsto A/D$:

$$\pi_2:\mathcal{X}_{\mathfrak{c},\Gamma_0(\mathfrak{a})}(\epsilon)_a\to\mathcal{X}_{\mathfrak{ca}}(\epsilon),\quad (A,\iota,\lambda,\mu_N,D)\to(A':=A/D,\iota',\lambda',\mu'_N).$$

Now let $n\in\mathbb{Z}_{\geq 1}\cup\{\infty\}$. Then it is clear that the morphisms π_1 and π_2 extend uniquely when we add level structures at p: more precisely, we obtain a natural commutative diagram

$$\begin{array}{ccccc}
\mathcal{X}_{\mathfrak{ca},\Gamma(p^\infty)}(\epsilon)_a & \xleftarrow{\pi_{2,\infty}} & \mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)\cap\Gamma_0(\mathfrak{a})}(\epsilon)_a & \xrightarrow{\pi_{1,\infty}} & \mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{\mathfrak{ca},\Gamma_0(p^n)}(\epsilon)_a & \xleftarrow{\pi_2} & \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{a})}(\epsilon)_a & \xrightarrow{\pi_1} & \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)}(\epsilon)_a,
\end{array} \tag{10.1}$$

where $\pi_{2,\infty}$ sends $\alpha : \mathcal{O}_p^2 \to T_pA^\vee$ to $\alpha' : \mathcal{O}_p^2 \xrightarrow{\alpha} T_pA^\vee \xrightarrow{(\varphi^\vee)^{-1}} T_pA'^\vee$, where the isomorphism $\varphi^\vee : T_pA'^\vee \to T_pA^\vee$ is the one induced from the prime-to-p isogeny $\varphi^\vee : A'^\vee = (A/D)^\vee \to A^\vee$.

Lemma 10.1. *There is a canonical isomorphism $\pi_1^*\omega_n^{\kappa,+} = \pi_2^*\omega_n^{\kappa,+}$ of sheaves on $\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{a})}(\epsilon)_a$.*

Proof. By diagram (10.1), we have

$$\pi_1^*\omega_n^{\kappa,+} = \{f \in \mathcal{O}^+_{\mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)\cap\Gamma_0(\mathfrak{a})}(\epsilon)_a} \mid \gamma^* f = \pi_{1,\infty}^*\kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma \in \Gamma_0(p^n)\}.$$

The same applies to $\pi_2^*\omega_n^{\kappa,+}$, so we are left to see that $\pi_{1,\infty}^*\kappa^{-1}(c\mathfrak{z}+d) = \pi_{2,\infty}^*\kappa^{-1}(c\mathfrak{z}+d)$. To see this, observe that $\varphi : A \to A/D$ induces an isomorphism of Hodge–Tate sequences

$$\begin{array}{ccccc} \mathcal{O}_p^2 & \xrightarrow{\alpha} & T_p(A/D)^\vee & \xrightarrow{\mathrm{HT}} & \omega_{A/D} \\ \| & & \downarrow{\scriptstyle \varphi^\vee} & & \downarrow{\scriptstyle \varphi^*} \\ \mathcal{O}_p^2 & \xrightarrow{\alpha'} & T_pA^\vee & \xrightarrow{\mathrm{HT}} & \omega_A \end{array} \tag{10.2}$$

by definition of α. This shows that $\pi_{\mathrm{HT}} \circ \pi_{1,\infty} = \pi_{\mathrm{HT}} \circ \pi_{2,\infty}$, giving the desired equality. □

Definition 10.2. The $T_{\mathfrak{a}}$-operator is defined as the composition

$$M_\kappa^{G^*,+}(\Gamma_0(p^n), \mu_N, \epsilon, \mathfrak{c}\mathfrak{a}) = \Gamma(\mathcal{X}_{\mathfrak{c}\mathfrak{a},\Gamma_0(p^n)}(\epsilon)_a, \omega_n^{\kappa,+}) \to \Gamma(\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{a})}(\epsilon)_a, \pi_2^*\omega_n^{\kappa,+})$$

$$\xrightarrow{\sim} \Gamma(\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{a})}(\epsilon)_a, \pi_1^*\omega_n^{\kappa,+}) \xrightarrow{\frac{1}{q_{\mathfrak{a}}}\mathrm{Tr}_{\pi_1}} \Gamma(\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)}(\epsilon)_a, \omega_n^{\kappa,+}) = M_\kappa^{G^*,+}(\Gamma_0(p^n), \mu_N, \epsilon, \mathfrak{c}).$$

where Tr_{π_1} is the trace of the finite locally free map π_1, and where $q_{\mathfrak{a}} := |\mathcal{O}_F/\mathfrak{a}|$.

10.2. **The $U_{\mathfrak{p}}$-operators.** Let $n \in \mathbb{Z}_{\geq 1}$ and $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ above p of ramification index e. Set $l := ne+1$. For the definition of the $U_{\mathfrak{p}}$-operator, we then use the moduli space $\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a \to \mathcal{X}_{\mathfrak{c}}(\epsilon)$ which relatively represents the data of an anticanonical $\mathcal{O}_F$-submodule scheme $C \subseteq A[p^n]$ étale locally isomorphic to $\mathcal{O}_F/p^n\mathcal{O}_F$ together with an $\mathcal{O}_F$-submodule scheme $D \subseteq A[\mathfrak{p}^l]$ étale locally isomorphic to $\mathcal{O}_F/\mathfrak{p}^l$ such that $C[\mathfrak{p}^{en}] = D[\mathfrak{p}^{en}]$. In particular, D is then anticanonical. There is a forgetful map $\pi_1 : \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a \to \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)}(\epsilon)_a$ which is finite flat of degree $q_{\mathfrak{p}} := |\mathcal{O}_F/\mathfrak{p}|$. There is also a second map

$$\begin{aligned} \pi_2 : \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a &\to \mathcal{X}_{\mathfrak{c}\mathfrak{p},\Gamma_0(p^n)}(\epsilon)_a \\ (A, \iota, \lambda, \mu_N, C, D) &\mapsto (A' := A/D[\mathfrak{p}], \iota', \lambda', \mu_N', C' := (C+D)/D[\mathfrak{p}]) \end{aligned}$$

where $(A', \iota', \lambda', \mu_N')$ is like in Lem. 8.22 and where $C + D \subseteq A[p^{n+1}]$ is the submodule scheme generated by C and D. Then C' is étale locally isomorphic to $\mathcal{O}_F/p^n\mathcal{O}_F$. We note that this map is not surjective, and the image is already contained in an open subspace that can be described using the partial Hasse invariant at $\mathfrak{p}$ (for example, if $F = \mathbb{Q}$, then it lands in $\mathcal{X}_{p\mathfrak{c},\Gamma_0(p^n)}(p^{-1}\epsilon)_a$).

Let us now fix any uniformiser $\varpi \in \mathcal{O}_p$ such that $\varpi\mathcal{O}_p = \mathfrak{p}\mathcal{O}_p$ and let $u_{\mathfrak{p}} := \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(\mathbb{Q}_p)$. Then letting $u_{\mathfrak{p}}$ act in terms of the $G(\mathbb{Q}_p)$-action, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}_{\mathfrak{c}\mathfrak{p},\Gamma(p^\infty)}(\epsilon)_a & \xleftarrow{u_{\mathfrak{p}}} & \mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a & = & \mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a \\ \downarrow{\scriptstyle q} & & \downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X}_{\mathfrak{c}\mathfrak{p},\Gamma_0(p^n)}(\epsilon)_a & \xleftarrow{\pi_2} & \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a & \xrightarrow{\pi_1} & \mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)}(\epsilon)_a. \end{array} \tag{10.3}$$

Lemma 10.3. *The action of $u_{\mathfrak{p}}$ defines a map $\pi_2^*\omega_n^{\kappa,+} \to \pi_1^*\omega_n^{\kappa,+}$ of invertible sheaves on $\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a$. It is independent of the choice of the uniformiser ϖ.*

Proof. By diagram (10.3), we have

$$\pi_1^*\omega_n^{\kappa,+} = \{f \in \mathcal{O}^+_{\mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a} \mid \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d)f \text{ for all } \gamma \in \Gamma_0(p^n) \cap \Gamma_0(\mathfrak{p}^l)\}.$$

We claim that $u_{\mathfrak{p}}^*\omega_n^{\kappa,+}$ admits the same description. To see this, we first recall that the action of $u_{\mathfrak{p}}$ is equivariant with respect to the morphism $\Gamma_0(p^n) \cap \Gamma_0(\mathfrak{p}^l) \to \Gamma_0(p^n)$ given by conjugation by $u_{\mathfrak{p}}$, namely $j : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & \varpi b \\ \varpi^{-1}c & d \end{smallmatrix}\right)$. Second, we see from $G(\mathbb{Q}_p)$-equivariance of π_{HT} that $u_{\mathfrak{p}}^*\mathfrak{z} = \varpi\mathfrak{z}$. Consequently, $\kappa(\gamma, \mathfrak{z}) := \kappa(c\mathfrak{z}+d)$ is sent by $u_{\mathfrak{p}}$ to $(\gamma, \mathfrak{z}) \mapsto \kappa(j(\gamma), \varpi\mathfrak{z}) = \kappa(\varpi^{-1}\varpi c\mathfrak{z} + d) = \kappa(c\mathfrak{z}+d)$. This together with the fact that shows that $\mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a \to X_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a$ is a pro-étale $\Gamma_0(p^n) \cap \Gamma_0(\mathfrak{p}^l)$-torsor shows that $u_{\mathfrak{p}}^*\omega_n^{\kappa,+}$ has the desired form. □

Definition 10.4. The $U_{\mathfrak{p}}$-operator is defined as the composition

$$M_\kappa^{G^*}(\Gamma_0(p^n),\mu_N,\epsilon,\mathfrak{cp}) = \Gamma(\mathcal{X}_{\mathfrak{cp},\Gamma_0(p^n)}(\epsilon)_a,\omega_n^\kappa) \to \Gamma(\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a,\pi_2^*\omega_n^\kappa)$$
$$\longrightarrow \Gamma(\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a,\pi_1^*\omega_n^\kappa) \xrightarrow{\frac{1}{q_{\mathfrak{p}}}\mathrm{Tr}_{\pi_1}} \Gamma(\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)}(\epsilon)_a,\omega_n^\kappa) = M_\kappa^{G^*}(\Gamma_0(p^n),\mu_N,\epsilon,\mathfrak{c}).$$

10.3. **The Hecke action on arithmetic Hilbert modular forms.** For $\mathfrak{c}$ a polarisation ideal, we let $[\mathfrak{c}]$ denote its class in the narrow class group. One can then define Hecke operators on the spaces of overconvergent Hilbert modular forms for G of the form

$$T_{\mathfrak{a}} : M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,[\mathfrak{ca}]) \to M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,[\mathfrak{c}])$$
$$U_{\mathfrak{p}} : M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,[\mathfrak{cp}]) \to M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon,[\mathfrak{c}])$$

by taking Δ-invariants of the operators defined in the last section. Alternatively, one can define these operators more directly and without any reference to G^* based on Defns. 9.1 and Defn. 9.8; copying the definitions for G^* and replacing $\mathcal{X}$ by $\mathcal{X}_G$ throughout. The proofs go through without change. The natural morphisms of Hecke correspondences over the map $\mathcal{X} \to \mathcal{X}_G$ shows that the operators thus defined coincide with the ones obtained via G^*.

It is clear from either definition that the Hecke operators commute with the polarisation action from Defn. 9.10. Consequently, they induce a Hecke action on $M_\kappa^G(\Gamma_0(p^n),\mu_N,\epsilon)$.

Remark 10.5. Via the Koecher principle the Hecke operators also extend to the boundary. Moreover, the subspaces of cusp forms will be preserved by the action of Hecke operators.

Remark 10.6. As defined, the Hecke operators for G^* are canonical. If one fixes a set of representatives $(\mathfrak{c}_i)$ of the narrow class group and considers the Hecke operators as mapping between the spaces for these fixed $\mathfrak{c}_i$, then the Hecke operators for G^* become non-canonical, depending on the choice of representatives (cf. [AIP16b, Section 4.3]). For G, the operators remain canonical as picking different representatives does not affect our polarisation class. In particular, we get commuting Hecke operators on $M_k^G(\Gamma_0(p^n),\mu_N,\epsilon)$.

Lastly, one can check directly that for G^* or G the Hecke operator $U_p = \prod_{\mathfrak{p}|p} U_{\mathfrak{p}}^{e_i}$ is a compact operator. Alternatively, this follows from Prop. 10.8 together with [AIP16b, Lem. 3.27].

Remark 10.7. It is clear from the definition that the tame Hecke operators preserve the integral spaces of overconvergent forms, while for $U_{\mathfrak{p}}$ this is in general only true after renormalisation.

10.4. **Hecke-equivariance of the comparison.** We can now finish the proof of Thms. 7.14 and 9.12 by proving that the comparison isomorphisms are Hecke equivariant.

Proposition 10.8. *The isomorphisms $\omega_{G^*,n}^{\kappa,+} \xrightarrow{\sim} \omega_{G^*,\mathrm{AIP},n}^{\kappa,+}$ and $\omega_{G,n}^{\kappa,+} \xrightarrow{\sim} \omega_{G,\mathrm{AIP},n}^{\kappa,+}$ are Hecke equivariant on global sections.*

Proof. We consider the case of G^*, the case of G follows from this. As in the proof of Thm. 9.12, we can assume that κ has image in $\mathcal{W}_k^*$ for some $k > 0$. It is clear from comparing Definitions 10.2 and 10.4 to the definition in [AIP16a, §8.5] that it suffices to prove that the isomorphism $\pi_2^*\omega_{G^*,n}^\kappa = \pi_1^*\omega_{G^*,n}^\kappa$ from Lem. 10.1 and 10.3 is identified with the isomorphisms $\pi_2^*\omega_{G^*,n,\mathrm{AIP}}^\kappa = \pi_1^*\omega_{G^*,n,\mathrm{AIP}}^\kappa$ from [AIP16a, Lem. 8.5] under the comparison isomorphism. For this it suffices to see that the comparison map $\mathfrak{s}$ from §7.2 induces a morphism of Hecke correspondences

$$\begin{array}{ccccc} \mathcal{X}_{\mathfrak{cp},\Gamma(p^\infty)}(\epsilon)_a & \xleftarrow{\pi_{2,\infty}} & \mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a & \xrightarrow{\pi_{1,\infty}} & \mathcal{X}_{\mathfrak{c},\Gamma(p^\infty)}(\epsilon)_a \\ \downarrow{\scriptstyle\mathfrak{s}} & & \downarrow{\scriptstyle\mathfrak{s}} & & \downarrow{\scriptstyle\mathfrak{s}} \\ \mathcal{F}_m(\epsilon) & \xleftarrow{\pi_2} & \mathcal{F}_{m,\Gamma_0(\mathfrak{a})}(\epsilon) & \xrightarrow{\pi_1} & \mathcal{F}_m(\epsilon) \end{array} \tag{10.4}$$

where $\mathcal{F}_{m,\Gamma_0(\mathfrak{a})}(\epsilon)$ is the pullback along $\pi_1 : \mathcal{X}_{\mathfrak{c},\Gamma_0(\mathfrak{a})}(\epsilon) \to \mathcal{X}_{\mathfrak{c}}(\epsilon)$ of the Andreatta–Iovita–Pilloni torsor, and where π_2 is induced by the map $\omega_{A/D} \to \omega_A$. Commutativity of the right square is clear. In terms of moduli, commutativity of the left square is now precisely that of diagram (10.2).

For the $U_{\mathfrak{p}}$-operator, the top left map in diagram is replaced by the action of $u_{\mathfrak{p}} = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let A be the universal abelian variety over $\mathcal{X}_{\mathfrak{c},\Gamma_0(p^n)\cap\Gamma_0(\mathfrak{p}^l)}(\epsilon)_a$ with its anticanonical subgroup $D \subseteq A[\mathfrak{p}^l]$, then the map $\pi_2^*\omega_{G^*,n,\mathrm{AIP}}^\kappa = \pi_1^*\omega_{G^*,n,\mathrm{AIP}}^\kappa$ is induced via the adjoint $\omega_{G^*,n,\mathrm{AIP}}^\kappa \to \pi_{2,*}\pi_1^*\omega_{G^*,n,\mathrm{AIP}}^\kappa$ obtained by restriction from the map $\varphi^* : \omega_{A/D[\mathfrak{p}]} \to \omega_A$ associated to the isogeny $A \to A/D[\mathfrak{p}]$.

On the other hand, the morphism $\omega^{\kappa}_{G^*,n} \to \pi_{2,*}\pi_1^*\omega^{\kappa}_{G^*,n}$ in Lem. 10.3 is given by restriction of the action of $u_{\mathfrak{p}}$. To prove that the comparison is equivariant for the $U_{\mathfrak{p}}$-operator, it thus suffices to prove that these two morphisms commute with the comparison morphism $\mathfrak{s}$.

Using Lem. 8.23 and the identity $u_{\mathfrak{p}}^{\vee}(1,0) = (1,0)$, we see that the following diagram commutes

$$\begin{array}{ccccccc} (1,0) & & \mathcal{O}_p^2 & \xrightarrow{\alpha} & T_pA^{\vee} & \xrightarrow{\mathrm{HT}} & \omega_A \\ \uparrow & & \| u_{\mathfrak{p}}^{\vee} & & \uparrow \varphi^{\vee} & & \uparrow \varphi^* \\ (1,0) & & \mathcal{O}_p^2 & \xrightarrow{\alpha'} & T_p(A/D[\mathfrak{p}])^{\vee} & \xrightarrow{\mathrm{HT}} & \omega_{A/D[\mathfrak{p}]}. \end{array}$$

This shows that also $\mathfrak{s} \circ u_{\mathfrak{p}} = \varphi^* \circ \mathfrak{s}$, and thus $U_{\mathfrak{p}}$ commutes with the comparison isomorphism. □

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