Local and global blow-downs of transport twistor space

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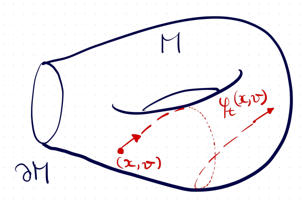
UNIVERSITÄT BONN

Let (M, g) be a Riemannian surface.

$$SM = \{(x, v) \in TM : g(v, v) = 1\}, \quad (\varphi_t) = \text{geodesic flow on } SM$$



Transport equations

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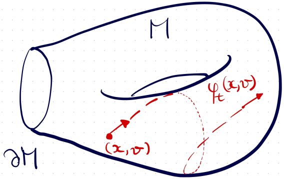

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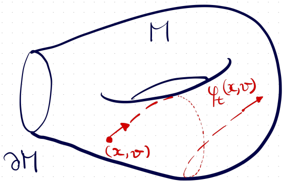

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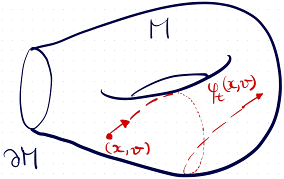

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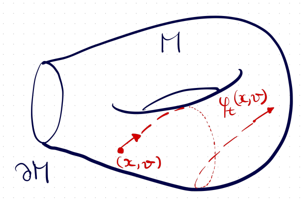

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Many geometric inverse problems are best understood via the transport equation. (cf. Paternain–Salo–Uhlmann: **GIP in 2D**, 2023)

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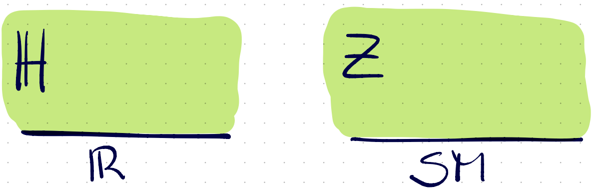

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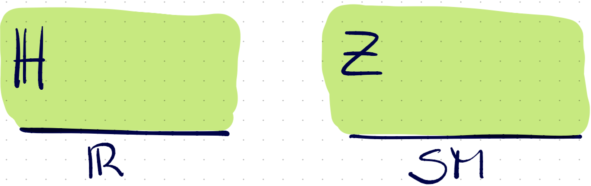

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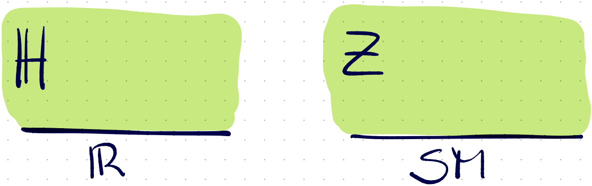

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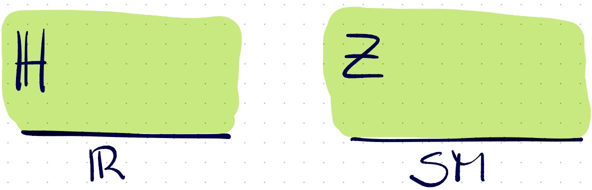

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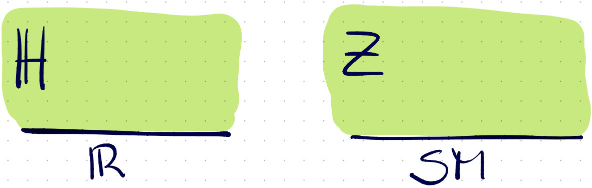

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- **Tool?**

Transport twistor space – Construction

Let $(M, g) = (\mathbb{D}, e^{2\sigma}|dz|^2)$ where $\sigma \in C^\infty(\mathbb{D}, \mathbb{R})$. Let

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Definition

On Z we define the complex vector field

$$\Xi_\sigma = e^{-\sigma}\left(\mu^2\cdot\partial_z + \partial_{\bar{z}} + \left(\mu^2\partial_z\sigma - \partial_{\bar{z}}\sigma\right)\cdot(\bar{\mu}\partial_{\bar{\mu}} + \mu\partial_\mu)\right) \in C^\infty(Z, T_{\mathbb{C}}Z)$$

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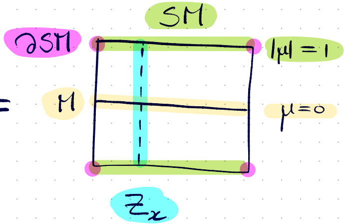
The **transport twistor space** of $(\mathbb{D}, e^{2\sigma}|dz|^2)$ is the (degenerate) complex surface

$$\boxed{(Z, \mathscr{D}_\sigma), \quad \mathscr{D}_\sigma = \mathrm{span}_{\mathbb{C}}(\Xi_\sigma, \partial_{\bar\mu}) \subset T_{\mathbb{C}}Z}$$

Transport twistor space – Characterisation

$Z =$ 

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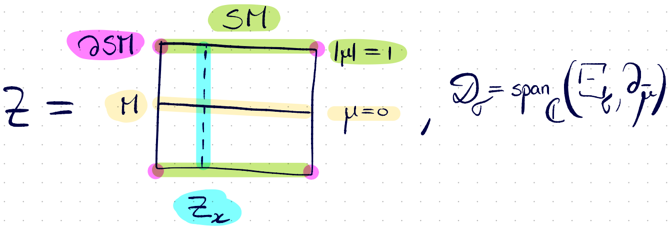

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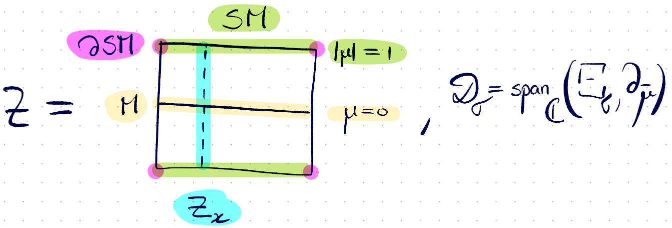

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  2. $[\mathscr{D}_\sigma, \mathscr{D}_\sigma] \subset \mathscr{D}_\sigma$ *(involutivity)*
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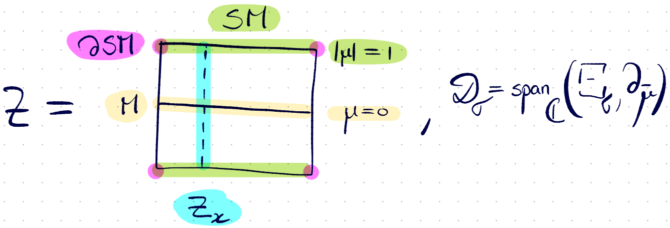

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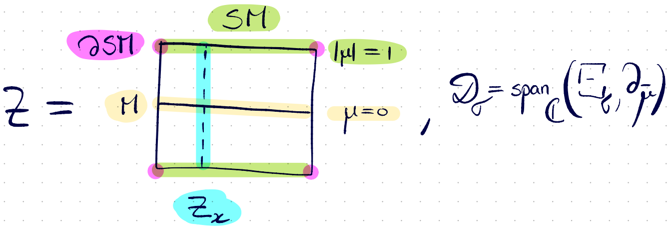

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- The interior Z° is a classical complex surface with $T^{0,1}Z^\circ = \mathscr{D}|_{Z^\circ}$
- Invariantly defined for **every oriented Riemannian surface** (M, g).

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[1] B.-Paternain, The transport Oka-Grauert principle for simple surfaces. JEP 2023

[2] B.-Lefeuvre-Paternain, Invariant distributions and the TTS of closed surfaces. Preprint 2023

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- **Next:** How to define such β-maps for other geometries? $\rightsquigarrow$ [3]

[3] B.-Monard-Paternain, Local & global blow downs of TTS, Preprint 2023+ϵ

Global blow-down maps

Theorem (Constant curvature disks)

If $(M,g) = (\mathbb{D}, e^{2\sigma_\kappa}|dz|^2)$ *with* $\sigma_\kappa = -\log\big(1+\kappa|z|^2\big)$ $(|\kappa|<1)$, *then*

$$\beta\colon Z \to \mathbb{C}^2, \quad \beta(z,\mu) = \left(\frac{z-\mu^2\bar{z}}{1+\kappa\bar{z}^2\mu^2}, \mu\frac{1+\kappa|z|^2}{1+\kappa\bar{z}^2\mu^2}\right)$$

has **holomorphic blow-down structure (HBS)**.

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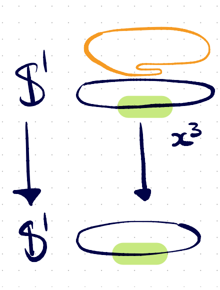
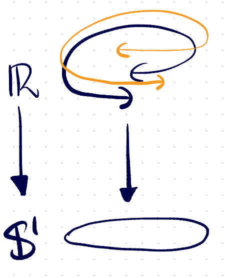


(a) Topological embeddings are not open.



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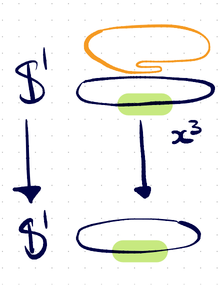

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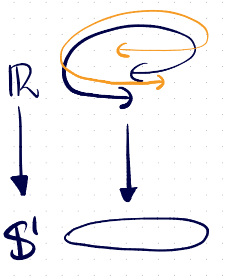

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Theorem (Openness under simultaneous perturbations)

Suppose $\beta_0\colon (Z, \mathscr{D}_{\sigma_0}, \Omega_{\sigma_0}) \to \mathbb{C}^2$ *has* **HBS** *and*

$$\sigma \approx \sigma_0, \ \beta \approx \beta_0 \ \ (\textit{in } C^\infty\textit{-topology}), \quad d\beta(\mathscr{D}_\sigma) = 0.$$

Then also $\beta\colon (Z, \mathscr{D}_\sigma, \underline{\Omega}_\sigma) \to \mathbb{C}^2$ *has* **HBS**.

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Theorem (Classical Newlander–Nirenberg theorem)

Let X be a $2n$-manifold and $\mathscr{D} \subset T_{\mathbb{C}}X$ an involutive distribution of rank n. For every point $p \in X$ with $\mathscr{D} \cap \overline{\mathscr{D}}(p) = 0$ there exists a neighbourhood $U \subset X$ and an embedding $\beta \colon U \to \mathbb{C}^n$ with $d\beta(\mathscr{D}) = 0$.

Thank you for your attention.